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Bosonic representations of Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ with $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$

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Abstract. On the basis of the '$RTT = TTR$' formalism, we introduce the quantum double of the Yangian $Y_{\hbar}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$ with a central extension. The Gauss decomposition of the *T*-matrices gives us the so-called Drinfel'd generators. Using these generators, we present some examples of both finite- and infinite-dimensional representations that are quite natural deformations of their corresponding affine counterpart.

1. Introduction

In the last few decades, the quantum inverse scattering method (QISM), initiated by Faddeev and co-workers, has been studied extensively and has produced rich structures in both physics and mathematics. The quantum algebras called the quantized universal enveloping algebra $U_q(\mathfrak{g})$ and the Yangian $Y_{\hbar}(\mathfrak{g})$ are some of the most important fruits inspired by the QISM. They have unexpected connections with such, at first sight unrelated, parts of mathematics as the construction of knot invariants, the geometric interpretation of a certain class of special functions and the representation theory of algebraic groups in the characteristic p. Of course they also have many nice applications in theoretical physics such as quantum field theory and statistical mechanics. As is well known, $U_q(\mathfrak{g})$ describes some features of conformal field theory. One can solve lattice models, like the spin-$\frac{1}{2}$ XXZ model, as an application of the representation theory of $U_q(\hat{\mathfrak{sl}}_2)$. The quantum affine algebra $U_q(\hat{\mathfrak{sl}}_2)$ is the q-deformation of the enveloping algebra $U(\hat{\mathfrak{sl}}_2)$. The Yangian $Y_{\hbar}(\mathfrak{g})$ is also related to conformal field theory. Lattice models such as the Haldane–Shastry model are known to possess $Y_{\hbar}(\mathfrak{sl}_2)$-symmetry. The Yangian $Y_{\hbar}(\mathfrak{sl}_2)$ is the $\hbar$-deformation of the enveloping algebra $U(\mathfrak{sl}_2[t])$. The quantum double [Dr1] of the $Y_{\hbar}(\mathfrak{g})$, which we shall refer to as Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$, seems to play important roles in massive field theory [BL, LS, S]. In these works, the Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ is the $\hbar$-deformation of the universal enveloping algebra of the loop algebra $\mathfrak{g}[t, t^{-1}]$ for $\mathfrak{g} = \mathfrak{sl}_2$, *without central extension*. In view of the lattice models, like the spin-$\frac{1}{2}$ XXX model of infinite chains, it seems necessary to construct the Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ with a central extension. In our previous paper [IK], we defined the Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ with a central extension for $\mathfrak{g} = \mathfrak{gl}_2$ or $\mathfrak{sl}_2$. The present paper is a higher rank generalization of it. Our attempt here is to explain the background of the construction and to consider the representation theory. We also summarize some formulae related to our calculations which seem well known to the specialists but have never appeared in the literature. The main topics treated in this paper are as follows.

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1. Yangian double

The Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ has been introduced into the literature in terms of Chevalley generators [LS], the $T^{\pm}$-matrix [BL] for $\mathfrak{g} = \mathfrak{sl}_2$ and Drinfel'd generators [KT] for a simple finite-dimensional Lie algebra $\mathfrak{g}$. Here we construct $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$ by means of the QISM [BL, RS, RTF]. Namely, let $R(u)$ be the Yang R-matrix. The algebra $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ is defined through quadratic relations of the form

$$R(u-v)(T^{\pm}(u)\otimes \mathrm{id})(\mathrm{id}\otimes T^{\pm}(v)) = (\mathrm{id}\otimes T^{\pm}(v))(T^{\pm}(u)\otimes \mathrm{id})R(u-v)$$

$$R(u-v-\tfrac{1}{2}\hbar c)(T^{+}(u)\otimes \mathrm{id})(\mathrm{id}\otimes T^{-}(v)) = (\mathrm{id}\otimes T^{-}(v))(T^{+}(u)\otimes \mathrm{id})R(u-v+\tfrac{1}{2}\hbar c)$$

where c is a central element of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$. The $T^{\pm}$-matrices $T^{\pm}(u) = (t^{\pm}_{ij}(u))_{1\leqslant i,j\leqslant N}$ are expanded as

$$t^{+}_{ij}(u) = \delta_{ij} - \hbar\sum_{k\geqslant 0} t^{k}_{ij}u^{-k-1} \qquad t^{-}_{ij}(u) = \delta_{ij} + \hbar\sum_{k<0} t^{k}_{ij}u^{-k-1}.$$

Just as in the case of $U_q(\hat{\mathfrak{gl}}_n)$ [DF], we consider the Gauss decomposition of the $T^{\pm}$-matrix (theorem 3.2) and obtain the Drinfel'd generators of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ (theorem 3.3). We define $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ as a certain subalgebra of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ and show that our Drinfel'd generators recover the results obtained in [KT] at level 0 (corollary 3.5). We also introduce another subalgebra of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ which we call the Heisenberg subalgebra.

2. Representation theory

Here we investigate several examples. The main tool here is the Drinfel'd generators.

Finite-dimensional representations. At $c = 0$, the Heisenberg subalgebra of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ becomes the centre of it. So we will concentrate on $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ case without loss of generality. From the commutation relations of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ at level 0 (corollary 3.5), we expect that the analogue of the classification theorem of irreducible finite-dimensional representations holds just as in the case of Yangian $Y_{\hbar}(\mathfrak{g})$ [Dr3]. We present some examples which support our conjecture. All of them are ones that we call evaluation modules.

Infinite-dimensional representations. Unfortunately we have no proper definition of highest-weight modules due to the lack of the triangular decomposition of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$. Here we realize level 1 $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$-modules on the boson Fock space $\mathcal{F}_{i,s}$ $(0\leqslant i\leqslant N-1, s\in\mathbb{C})$ (theorem 4.5). Let V_u be an N-dimensional evaluation modules of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$. The vertex operators are intertwiners of the form

$$\Phi^{(i,i+1)}(u) : \mathcal{F}_{i+1,s} \longrightarrow \mathcal{F}_{i,s-1}\otimes V_u$$

$$\Psi^{(i,i+1)}(u) : \mathcal{F}_{i+1,s} \longrightarrow V_u\otimes \mathcal{F}_{i,s-1}.$$

We also give the bosonization of the vertex operators (theorem 4.6). For the $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ case, we construct level 1 modules on the boson Fock space $\mathcal{F}_i$ $(0\leqslant i\leqslant N-1)$ (theorem 4.7) whose quantum affine versions are obtained in [FJ]. We should mention that every field defined above makes sense as a formal series in $\hbar$. Moreover, we also construct vertex operators for $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$, in which case the Fourier components lose their meaning (theorem 4.8). More precisely, those formulae makes sense only as an asymptotic series.

The text is organized as follows. In section 2 we recall the definition of Yangian $Y_{\hbar}(\mathfrak{g})$. We also mention the other set of generators and the isomorphism between them. The theory of finite-dimensional $Y_{\hbar}(\mathfrak{g})$-modules is also reviewed and one example is given. In section 3 we define $\mathcal{D}Y_\hbar(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$. We rewrite the commutation relations in terms of Drinfel'd generators. In section 4 we present a conjecture for finite-dimensional $\mathcal{D}Y_\hbar(\mathfrak{sl}_N)$-modules together with a few examples. As for infinite-dimensional representations, we construct level 1 modules and vertex operators directly via bosonization. Section 5 contains discussions and remarks. For the reader's convenience, we also include two appendices. In appendix A we give a brief review of quantum groups, in particular the universal $\mathcal{R}$- and L-operators. In appendix B we collect some formulae for T-matrices.

Let us mention that the author got two papers [K1, K2] when he was preparing this paper. The central extension of $\mathcal{D}Y_\hbar(\mathfrak{sl}_2)$ is introduced in [K1] which has some overlap with [IK]. The bosonizations of level 1 $\mathcal{D}Y_\hbar(\mathfrak{sl}_2)$-module and the vertex operators among them are obtained in [K2]. Here we introduce the Yangian double $\mathcal{D}Y_\hbar(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$ with a centre and obtain the bosonization of level 1 $\mathcal{D}Y_\hbar(\mathfrak{g})$-module and the vertex operators among them.

2. Review of the Yangian $Y_\hbar(\mathfrak{g})$

In this section we collect some known facts about Yangians, including representation theory.

2.1. Yangian $Y_\hbar(\mathfrak{g})$

Here we present two different realizations of $Y_\hbar(\mathfrak{g})$ for a simple finite-dimensional Lie algebra $\mathfrak{g}$. In addition, for $\mathfrak{g} = \mathfrak{sl}_N$, another realization called the T-matrix is known [Dr1, Dr2], and we make some comments on it.

Set $\mathcal{A} = \mathbb{C}[[\hbar]]$. Let $\mathfrak{g}$ be a simple finite-dimensional Lie algebra and $\{\alpha_1, \alpha_2, \cdots, \alpha_n\}$ the set of simple roots. Fix a standard non-degenerate symmetric invariant bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$. For each positive root α of $\mathfrak{g}$, choose root vectors $x^\pm_\alpha$ in $\pm\alpha$ root spaces such that $(x^+_\alpha, x^-_\alpha) = 1$ and set $h_\alpha = [x^+_\alpha, x^-_\alpha]$. We denote the Cartan matrix of $\mathfrak{g}$ by $A = (a_{ij})$.

Let $\{I_p\}$ be any orthonormal basis of $\mathfrak{g}$ with respect to the inner product $(\cdot,\cdot)$.

Definition 2.1 ([Dr3]). The Yangian $Y_\hbar(\mathfrak{g})$ is a topological Hopf algebra over $\mathcal{A}$ generated by $\mathfrak{g}$ and elements $J(x)$, $x \in \mathfrak{g}$, with relations

$$J(ax+by) = aJ(x) + bJ(y) \qquad a, b \in \mathcal{A} \qquad [x, J(y)] = J([x,y])$$

$$\begin{aligned}
&[J(x), J([y,z])] + [J(y), J([z,x])] + [J(z), J([x,y])] \\
&\qquad = \hbar^2 \sum_{p,q,r} ([x, I_p], [[y, I_q], [z, I_r]])\{I_p, I_q, I_r\}
\end{aligned}$$

$$\begin{aligned}
&[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]] \\
&\qquad = \hbar^2 \sum_{p,q,r} \Big(([x, I_p], [[y, I_q], [[z,w], I_r]]) \\
&\qquad\quad + ([z, I_p], [[w, I_q], [[x,y], I_r]])\Big)\{I_p, I_r, J(I_r)\}
\end{aligned}$$

where $\{\cdot,\cdot,\cdot\}$ denotes the symmetrization

$$\{x_1, x_2, x_3\} = \frac{1}{24} \sum_{\sigma \in \mathfrak{S}_3} x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)}.$$

The comultiplication of $Y_\hbar(\mathfrak{g})$ is given by

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$\Delta(J(x)) = J(x) \otimes 1 + 1 \otimes J(x) + \tfrac{1}{2}\hbar[x \otimes 1, \Omega]$$

where Ω stands for the Casimir element of $\mathfrak{g} \otimes \mathfrak{g}$.

Drinfel'd [Dr3] has shown that there are so-called Drinfel'd generators of $Y_{\hbar}(\mathfrak{g})$. To be precise, the following theorem holds.

Theorem 2.1 ([Dr3]). The Yangian $Y_{\hbar}(\mathfrak{g})$ is isomorphic to the algebra generated by the elements $\{\xi^{\pm}_{ik}, \kappa_{ik} | 1 \leqslant i \leqslant n\ ,\ k \in \mathbb{Z}_{\geqslant 0}\}$ subject to the relations

$$[\kappa_{ik}, \kappa_{jl}] = 0 \qquad [\kappa_{i0}, \xi^{\pm}_{jl}] = \pm(\alpha_i, \alpha_j)\xi^{\pm}_{jl} \qquad [\xi^{+}_{ik}, \xi^{-}_{jl}] = \delta_{ij}\kappa_{ik+l}$$

$$[\kappa_{ik+1}, \xi^{\pm}_{jl}] - [\kappa_{ik}, \xi^{\pm}_{jl+1}] = \pm\tfrac{1}{2}(\alpha_i, \alpha_j)\hbar[\kappa_{ik}, \xi^{\pm}_{jl}]_{+}$$

$$[\xi^{\pm}_{ik+1}, \xi^{\pm}_{jl}] - [\xi^{\pm}_{ik}, \xi^{\pm}_{jl+1}] = \pm\tfrac{1}{2}(\alpha_i, \alpha_j)\hbar[\xi^{\pm}_{ik}, \xi^{\pm}_{jl}]_{+}$$

$$\sum_{\sigma \in \mathfrak{S}_m} [\xi^{\pm}_{ik_{\sigma(1)}}, [\cdots, [\xi^{\pm}_{ik_{\sigma(m)}}, \xi^{\pm}_{jl}] = 0 \qquad \text{for} \quad i \neq j$$

where we set $m = 1 - a_{ij}$ and $[x, y]_{+} = xy + yx$ for $x, y \in Y_{\hbar}(\mathfrak{g})$. The isomorphism ϕ between two presentations is given by

$$\phi(h_i) = \kappa_{i0} \qquad \phi(x^{\pm}_i) = \xi^{\pm}_{i0}$$

$$\phi(J(h_i)) = \kappa_{i1} + \hbar\phi(v_i) \qquad \phi(J(x^{\pm}_i)) = \xi^{\pm}_{i1} + \hbar\phi(w^{\pm}_i)$$

where we set $h_i = h_{\alpha_i}, x^{\pm}_i = x^{\pm}_{\alpha_i}$ and

$$v_i = \frac{1}{4}\sum_{\alpha \succ 0}(\alpha, \alpha_i)(x^{+}_{\alpha}x^{-}_{\alpha} + x^{-}_{\alpha}x^{+}_{\alpha}) - \frac{1}{2}{h_i}^2$$

$$w^{\pm}_i = \pm\frac{1}{4}\sum_{\alpha \succ 0}\{[x^{\pm}_i, x^{\pm}_{\alpha}]x^{\mp}_{\alpha} + x^{\mp}_{\alpha}[x^{\pm}_i, x^{\pm}_{\alpha}]\} - \frac{1}{4}(x^{\pm}_i h_i + h_i x^{\pm}_i).$$

For $\mathfrak{g} = \mathfrak{sl}_N$, we have another realization called T-matrix [Dr2, Dr3] as follows.

Let V be a rank N $\mathcal{A}$-free module and $\mathcal{P} \in \mathrm{End}(V \otimes V)$ be a permutation operator $\mathcal{P}v \otimes w = w \otimes v$ $(v, w \in V)$. Consider Yang's R-matrix normalized as

$$R(u) = \frac{1}{u + \hbar}(uI + \hbar\mathcal{P}) \in \mathrm{End}(V \otimes V). \tag{1}$$

where $\hbar$ is expanded in positive powers. This R-matrix satisfies the following properties:

Yang–Baxter equation:

$$R_{12}(u - v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u - v)$$

Unitarity:

$$R_{12}(u)R_{21}(-u) = \mathrm{id}\,.$$

Here, if $R(u) = \sum a_i \otimes b_i$ with $a_i, b_i \in \mathrm{End}(V)$, then $R_{21}(u) = \sum b_i \otimes a_i$, $R_{13}(u) = \sum a_i \otimes 1 \otimes b_i$ etc.

Theorem 2.2 ([Dr3]). The Yangian $Y_{\hbar}(\mathfrak{sl}_N)$ is isomorphic to the algebra with generators $\{t^{k}_{i,j} | 1 \leqslant i, j \leqslant N\ ,\ k \in \mathbb{Z}_{\geqslant 0}\}$ and defining relations

$$R(u - v)\overset{1}{T}(u)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(u)R(u - v) \qquad q\text{-det}\,T(u) = 1.$$

Here

$$T(u) = (t_{ij}(u))_{1\leqslant i,j\leqslant N} \qquad t_{ij}(u) = \delta_{ij} - \hbar \sum_{k\in\mathbb{Z}_{\geqslant 0}} t_{ij}^k u^{-k-1}$$

$$\overset{1}{T}(u) = T(u) \otimes \mathrm{id} \qquad \overset{2}{T}(u) = \mathrm{id} \otimes T(u)$$

and q-det $T(u)$ is defined in proposition B.4. The comultiplication is given by

$$\Delta(t_{ij}(u)) = \sum_{k=1}^{N} t_{kj}(u) \otimes t_{ik}(u).$$

Roughly speaking, the isomorphism between the algebra generated by the Drinfel'd generators and the algebra presented above is given by the Gauss decomposition of the T-matrix (see section 3 and appendix B.2 for details).

2.2. Representation theory of $Y_{\hbar}(\mathfrak{g})$

In this subsection, we give a brief review on finite-dimensional representations of $Y_{\hbar}(\mathfrak{g})$. See [CP1, CP2] for detail.

Let $\boldsymbol{h} = \{h_{i,r}\}_{1\leqslant i\leqslant n, r\in\mathbb{Z}_{\geqslant 0}}$ be a subset of $\mathcal{A}$. A $Y_{\hbar}(\mathfrak{g})$-module V is called the *highest-weight module with highest weight* $\boldsymbol{h}$ if there exits a unique, up to scalar, non-zero vector $v \in V$ such that V is generated by v and

$$\kappa_{i,r} v = h_{i,r} v \qquad \xi_{i,r}^{+} v = 0 \qquad 1 \leqslant \forall\ i \leqslant n \quad \forall\ r \in \mathbb{Z}_{\geqslant 0}.$$

It is known that every irreducible finite-dimensional $Y_{\hbar}(\mathfrak{g})$-module V is highest-weight module. Let us denote the irreducible highest weight $Y_{\hbar}(\mathfrak{g})$-module with highest weight $\boldsymbol{h}$ by $V(\boldsymbol{h})$. The criterion of the finite-dimensionality of $V(\boldsymbol{h})$ is known.

Theorem 2.3 ([Dr3]). The irreducible $Y_{\hbar}(\mathfrak{g})$-module $V(\boldsymbol{h})$ of highest weight $\boldsymbol{h}$ is finite dimensional if and only if there exist monic polynomials $P_i(v) \in \mathcal{A}[v]$ $1 \leqslant i \leqslant n$ such that

$$\frac{P_i(v + \frac{1}{2}(\alpha_i, \alpha_i)\hbar)}{P_i(v)} = 1 + \hbar \sum_{r=0}^{\infty} h_{i,r} v^{-r-1}$$

in the sense that the right-hand side is the Laurent expansion of the left-hand side about $v = \infty$.

The polynomials $P_i(v)$ in this theorem are called Drinfel'd polynomials.

The preceding theorem suggests the following definition.

Definition 2.2 ([CP2]). We say that an irreducible finite-dimensional $Y_{\hbar}(\mathfrak{g})$-module is *fundamental* if its Drinfel'd polynomials are given by

$$P_j(v) = \begin{cases} v - u & j = i \\ 1 & j \neq i \end{cases}$$

for some $1 \leqslant i \leqslant n$.

Using the fact that the Drinfel'd polynomials of the tensor product of two highest-weight modules are the product of the Drinfel'd polynomials of two highest-weight modules, one proves the following.

Theorem 2.4 ([CP2]). Every irreducible finite-dimensional $Y_{\hbar}(\mathfrak{g})$-module is isomorphic to a subquotient of a tensor product of fundamental representations.

Here we present an example of fundamental representation of $Y_{\hbar}(\mathfrak{sl}_N)$ and the more general representations containing the first example.

Example 2.1. Set

$$V_u = V \otimes_{\mathcal{A}} \mathcal{A}[u] \qquad V = \oplus_{j=0}^{N-1} \mathcal{A} w_j.$$

$Y_{\hbar}(\mathfrak{sl}_N)$-module structure on V_u is defined via the following actions:

(i) $$\xi_{i,r}^{+} w_i = \left(u - \frac{N-1-i}{2}\hbar\right)^r w_{i-1} \qquad \xi_{i,r}^{+} w_j = 0 \quad j \neq i$$

(ii) $$\xi_{i,r}^{-} w_{i-1} = \left(u - \frac{N-1-i}{2}\hbar\right)^r w_i \qquad \xi_{i,r}^{-} w_j = 0 \quad j \neq i-1$$

(iii) $$\kappa_{i,r} w_{i-1} = \left(u - \frac{N-1-i}{2}\hbar\right)^r w_{i-1}$$

$$\kappa_{i,r} w_i = -\left(u - \frac{N-1-i}{2}\hbar\right)^r w_i \qquad \kappa_{i,r} w_j = 0 \quad j \neq i, i-1.$$

Note that one can regard u as either an indeterminate or an element of $\mathcal{A}$. In the latter case, the Drinfel'd polynomials of V_u are given by

$$P_1(v) = v - \left(u - \frac{N-2}{2}\hbar\right) \qquad P_i(v) = 1 \quad i \neq 1.$$

Let us fix $\mathfrak{g}$ to be a simple finite-dimensional Lie algebra of classical type and normalize the invariant bilinear form by the condition $(\beta, \beta) = 2$ (β: long root). The fundamental weights $\Lambda_i (1 \leqslant i \leqslant \operatorname{rank} \mathfrak{g})$ are chosen so as to satisfy

$$2\frac{(\Lambda_i, \alpha_j)}{(\alpha_j, \alpha_j)} = \delta_{i,j}.$$

Example 2.2. Let $V(\Lambda)$ be the irreducible highest weight $\mathfrak{g}$-module with highest weight Λ. Especially when Λ is of the form $m\Lambda_i$ with m being positive integer, it is known from [KR] if $\mathfrak{g}$ is of type A_l (B_l, C_l, D_l) and $1 \leqslant i \leqslant l$ ($i = 1, i = l, i = 1, l-1, l$) then $V(m\Lambda_i)$ can be made into $Y_{\hbar}(\mathfrak{g})$-module. Let $V_a(m\Lambda_i)$ be such $Y_{\hbar}(\mathfrak{g})$-module satisfying

(i) $V_a(m\Lambda_i) \cong V(m\Lambda_i)$ as a $\mathfrak{g}$-module

(ii) $J(x)|_{V_a(m\Lambda_i)} = ax|_{V_a(m\Lambda_i)} \qquad \forall\, x \in \mathfrak{g}$.

The Drinfel'd polynomials of $V_a(m\Lambda_i)$ are given by

$$P_i(v) = \prod_{k=1}^{m} \left\{u - \left(a - \left(\tfrac{1}{4}g - j + \tfrac{1}{2}m\right)\hbar\right)\right\} \qquad P_j(v) = 1 \quad j \neq i$$

where g is the dual Coxeter number.

3. The algebra $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$

Here we define a central extension of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$ following the method of [RS].

3.1. Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$

Let us choose Yang's R-matrix as in (1).

Definition 3.1. $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ is a topological Hopf algebra over $\mathcal{A}$ generated by $\{t_{ij}^k|1 \leqslant i, j \leqslant N\ ,\ k \in \mathbb{Z}\}$ and c. In terms of matrix generating series

$$T^{\pm}(u) = (t_{ij}^{\pm}(u))_{1\leqslant i,j\leqslant N}$$

$$t_{ij}^{+}(u) = \delta_{ij} - \hbar \sum_{k\in\mathbb{Z}_{\geqslant 0}} t_{ij}^k u^{-k-1} \qquad t_{ij}^{-}(u) = \delta_{ij} + \hbar \sum_{k\in\mathbb{Z}_{<0}} t_{ij}^k u^{-k-1}$$

the defining relations are given as follows:

$$[T^{\pm}(u), c] = 0$$

$$R(u-v)\overset{1}{T}{}^{\pm}(u)\overset{2}{T}{}^{\pm}(v) = \overset{2}{T}{}^{\pm}(v)\overset{1}{T}{}^{\pm}(u)R(u-v)$$

$$R(u_- - v_+)\overset{1}{T}{}^{+}(u)\overset{2}{T}{}^{-}(v) = \overset{2}{T}{}^{-}(v)\overset{1}{T}{}^{+}(u)R(u_+ - v_-).$$

Here

$$\overset{1}{T}(u) = T(u)\otimes \mathrm{id} \qquad \overset{2}{T}(u) = \mathrm{id}\otimes T(u)$$

$u_{\pm} = u \pm \frac{1}{4}\hbar c$ and similarly for v. Its coalgebra structure is defined as

$$\Delta(t_{ij}^{\pm}(u)) = \sum_{k=1}^{N} t_{kj}^{\pm}(u \pm \tfrac{1}{4}\hbar c_2) \otimes t_{ik}^{\pm}(u \mp \tfrac{1}{4}\hbar c_1)$$

$$\varepsilon(T^{\pm}(u)) = I \qquad S({}^tT^{\pm}(u)) = [{}^tT^{\pm}(u)]^{-1}$$

$$\Delta(c) = c\otimes 1 + 1\otimes c \qquad \varepsilon(c) = 0 \qquad S(c) = -c$$

where $c_1 = c\otimes 1$ and $c_2 = 1\otimes c$.

Note that the subalgebra generated by $\{t_{ij}^k|1 \leqslant i, j \leqslant N\ ,\ k \in \mathbb{Z}_{\geqslant 0}\}$ is $Y_{\hbar}(\mathfrak{gl}_N)$ [Dr2, Dr3] and the algebra $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ is the quantum double of $Y_{\hbar}(\mathfrak{gl}_N)$. Let us define the pairing $\langle\cdot,\cdot\rangle$ between $T^{\pm}(u)$ as follows (cf [RTF]).

$$\langle T^{+}(u), T^{-}(v)\rangle := \sum_{i,j,k,l} \langle t_{ij}^{+}(u), t_{kl}^{-}(v)\rangle E_{ij}\otimes E_{kl} = R(u-v).$$

It seems that the following theorem is well known to the specialists.

Theorem 3.1. The pairing $\langle\cdot,\cdot\rangle$ gives the Hopf pairing.

The crucial point of the theorem is its non-degeneracy. We could check the non-degeneracy for $N = 2$ directly. For the motivation of our choice, see appendix A.

3.2. Drinfel'd generators

We introduce the Drinfel'd generators of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ exactly in the same way as in [DF].

Theorem 3.2. $T^{\pm}(u)$ have the following unique decompositions:

$$T^{\pm}(u) = \begin{pmatrix} 1 & & & 0 \\ f^{\pm}_{2,1}(u) & \ddots & & \\ & \ddots & \ddots & \\ f^{\pm}_{N,1}(u) & & f^{\pm}_{N,N-1}(u) & 1 \end{pmatrix} \begin{pmatrix} k^{\pm}_1(u) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & k^{\pm}_N(u) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & e^{\pm}_{1,2}(u) & & e^{\pm}_{1,N}(u) \\ & \ddots & \ddots & \\ & & \ddots & e^{\pm}_{N-1,N}(u) \\ 0 & & & 1 \end{pmatrix}.$$

To prove this theorem, we have only to show that each component $f^{\pm}_{p,q}(u)$, $k^{\pm}_p(u)$, $e^{\pm}_{p,q}(u)$ is well-defined element of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)[[u^{\mp 1}]]$. From the explicit formulae of these elements in terms of quantum minors, given in appendix B.2, it immediately follows since our algebra $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ is $\hbar$-adically completed. Set

$$X^-_i(u) = f^+_{i+1,i}(u_+) - f^-_{i+1,i}(u_-)$$

$$X^+_i(u) = e^+_{i,i+1}(u_-) - e^-_{i,i+1}(u_+).$$

They satisfy the following commutation relations.

Theorem 3.3.

$$k^{\pm}_i(u)k^{\pm}_j(v) = k^{\pm}_j(v)k^{\pm}_i(u) \qquad k^+_i(u)k^-_i(v) = k^-_i(v)k^+_i(u)$$

$$\frac{u_{\mp} - v_{\pm}}{u_{\mp} - v_{\pm} + \hbar} k^{\mp}_i(v)^{-1} k^{\pm}_j(u) = \frac{u_{\pm} - v_{\mp}}{u_{\pm} - v_{\mp} + \hbar} k^{\pm}_j(u) k^{\mp}_i(v)^{-1} \qquad i > j$$

$$\begin{cases} k^{\pm}_i(u)^{-1} X^+_i(v) k^{\pm}_i(u) = \dfrac{u_{\pm} - v + \hbar}{u_{\pm} - v} X^+_i(v) \\ k^{\pm}_i(u) X^-_i(v) k^{\pm}_i(u)^{-1} = \dfrac{u_{\mp} - v + \hbar}{u_{\mp} - v} X^-_i(v) \end{cases}$$

$$\begin{cases} k^{\pm}_{i+1}(u)^{-1} X^+_i(v) k^{\pm}_{i+1}(u) = \dfrac{u_{\pm} - v - \hbar}{u_{\pm} - v} X^+_i(v) \\ k^{\pm}_{i+1}(u) X^-_i(v) k^{\pm}_{i+1}(u)^{-1} = \dfrac{u_{\mp} - v - \hbar}{u_{\mp} - v} X^-_i(v) \end{cases}$$

$$k^{\pm}_j(u)^{-1} X^+_i(v) k^{\pm}_j(u) = X^+_i(v) \qquad k^{\pm}_j(u) X^-_i(v) k^{\pm}_j(u)^{-1} = X^-_i(v) \qquad \text{otherwise}$$

$$(u - v \mp \hbar) X^{\pm}_i(u) X^{\pm}_i(v) = (u - v \pm \hbar) X^{\pm}_i(v) X^{\pm}_i(u)$$

$$(u - v + \hbar) X^+_i(u) X^+_{i+1}(v) = (u - v) X^+_{i+1}(v) X^+_i(u)$$

$$(u - v) X^-_i(u) X^-_{i+1}(v) = (u - v + \hbar) X^-_{i+1}(v) X^-_i(u)$$

$$X^{\pm}_i(u_1) X^{\pm}_i(u_2) X^{\pm}_j(v) - 2X^{\pm}_i(u_1) X^{\pm}_j(v) X^{\pm}_i(u_2) + X^{\pm}_j(v) X^{\pm}_i(u_1) X^{\pm}_i(u_2)$$
$$+ \{u_1 \leftrightarrow u_2\} = 0 \qquad |i - j| = 1$$

$$X_i^{\pm}(u)X_j^{\pm}(v) = X_j^{\pm}(v)X_i^{\pm}(u) \qquad |i-j|>1$$

$$[X_i^{+}(u), X_j^{-}(v)] = \hbar\delta_{ij}\left\{\delta(u_- - v_+)k_{i+1}^{+}(u_-)k_i^{+}(u_-)^{-1} - \delta(u_+ - v_-)k_{i+1}^{-}(v_-)k_i^{-}(v_-)^{-1}\right\}.$$

Here $\delta(u-v) = \sum_{k\in\mathbb{Z}} u^{-k-1}v^k$ is a delta function.

One can prove the above theorem in exactly the same way as in [DF] for the $U_q(\hat{\mathfrak{gl}}_n)$ case.

3.3. Two subalgebras

To decompose $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ into two subalgebras $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ and a Heisenberg subalgebra, we introduce the following currents:

$$H_i^{\pm}(u) = k_{i+1}^{\pm}(u+\tfrac{1}{2}\hbar i)k_i^{\pm}(u+\tfrac{1}{2}\hbar i)^{-1} \qquad K^{\pm}(u) = \prod_{i=1}^{N} k_i^{\pm}\left(u + \left(i - \frac{N+1}{2}\right)\hbar\right)$$

$$E_i(u) = \frac{1}{\hbar}X_i^{+}(u+\tfrac{1}{2}\hbar i) \qquad F_i(u) = \frac{1}{\hbar}X_i^{-}(u+\tfrac{1}{2}\hbar i).$$

We define $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ to be the subalgebra of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ generated by $H_i^{\pm}(u)$, $E_i(u)$, $F_i(u)$ and c. A Heisenberg subalgebra of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ generated by $K^{\pm}(u)$ commute with all of the elements of $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$. In fact we see that the formula $K^{\pm}(u) = q\text{-}\det.T^{\pm}(u)$ holds as a consequence of theorem B.15. (See appendix B.2 for the definition of $q\text{-}\det.T^{\pm}(u)$.) In terms of these generators, the above commutation relations can be rephrased as follows. Let $A = (a_{ij})$ be the Cartan matrix of the Lie algebra $\mathfrak{sl}_N$.

Corollary 3.4.

$$[H_i^{\pm}(u), H_j^{\pm}(v)] = 0$$

$$(u_{\mp} - v_{\pm} + \hbar B_{ij})(u_{\pm} - v_{\mp} - \hbar B_{ij})H_i^{\pm}(u)H_j^{\mp}(v)$$
$$= (u_{\mp} - v_{\pm} - \hbar B_{ij})(u_{\pm} - v_{\mp} + \hbar B_{ij})H_j^{\mp}(v)H_i^{\pm}(u)$$

$$[K^{\pm}(u), K^{\pm}(v)] = 0 \qquad f(u_- - v_+)K^{+}(u)K^{-}(v) = K^{-}(v)K^{+}(u)f(u_+ - v_-)$$

$$\begin{cases} H_i^{\pm}(u)^{-1}E_j(v)H_i^{\pm}(u) = \dfrac{u_{\pm} - v - \hbar B_{ij}}{u_{\pm} - v + \hbar B_{ij}}E_j(v) \\[2ex] H_i^{\pm}(u)F_j(v)H_i^{\pm}(u)^{-1} = \dfrac{u_{\mp} - v - \hbar B_{ij}}{u_{\mp} - v + \hbar B_{ij}}F_j(v) \end{cases}$$

$$[K^{\sigma}(u), H_i^{\pm}(v)] = [K^{\sigma}(u), E_i(v)] = [K^{\sigma}(u), F_i(v)] = 0 \qquad \forall\, \sigma = \pm \quad \forall\, i$$

$$(u - v - \hbar B_{ij})E_i(u)E_j(v) = (u - v + \hbar B_{ij})E_j(v)E_i(u)$$

$$(u - v + \hbar B_{ij})F_i(u)F_j(v) = (u - v - \hbar B_{ij})F_j(v)F_i(u)$$

$$\sum_{\sigma\in\mathfrak{S}_m}[F_i(u_{\sigma(1)}), [F_i(u_{\sigma(2)})\cdots, [F_i(u_{\sigma(m)}), F_j(v)]\cdots] = 0 \qquad i\neq j \quad m = 1 - a_{ij}$$

$$\sum_{\sigma\in\mathfrak{S}_m}[E_i(u_{\sigma(1)}), [E_i(u_{\sigma(2)})\cdots, [E_i(u_{\sigma(m)}), E_j(v)]\cdots] = 0 \qquad i\neq j \quad m = 1 - a_{ij}$$

$$[E_i(u), F_j(v)] = \frac{1}{\hbar}\delta_{ij}\left\{\delta(u_- - v_+)H_i^{+}(u_-) - \delta(u_+ - v_-)H_i^{-}(v_-)\right\}.$$

Here we have set $B_{ij} = \frac{1}{2}(\alpha_i, \alpha_j)$ and

$$f(u) = \prod_{j=1}^{N-1} \frac{u - j\overline{h}}{u + j\overline{h}}.$$

To compare with the known results at $c = 0$ [KT], let us write down the commutation relations componentwise. The Fourier components of the generating series $H_i^{\pm}(u)$, $E_i(u)$, $F_i(u)$ are of the following form:

$$H_i^+(u) = 1 + \overline{h} \sum_{k \geqslant 0} h_{ik} u^{-k-1} \qquad H_i^-(u) = 1 - \overline{h} \sum_{k<0} h_{ik} u^{-k-1}$$

$$E_i(u) = \sum_{k \in \mathbb{Z}} e_{ik} u^{-k-1} \qquad F_i(u) = \sum_{k \in \mathbb{Z}} f_{ik} u^{-k-1}.$$

For $c = 0$, the commutation relations of $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$ in terms of the above Fourier component look simple as follows.

Corollary 3.5.

$$[h_{ik}, h_{jl}] = 0 \qquad [h_{i0}, x_{jl}^{\pm}] = \pm 2B_{ij} x_{jl}^{\pm} \qquad [x_{ik}^+, x_{jl}^-] = \delta_{ij} h_{ik+l}$$

$$[h_{ik+1}, x_{jl}^{\pm}] - [h_{ik}, x_{jl+1}^{\pm}] = \pm \overline{h} B_{ij} [h_{ik}, x_{jl}^{\pm}]_+$$

$$[x_{ik+1}^{\pm}, x_{jl}^{\pm}] - [x_{ik}^{\pm}, x_{jl+1}^{\pm}] = \pm \overline{h} B_{ij} [x_{ik}^{\pm}, x_{jl}^{\pm}]_+$$

$$\sum_{\sigma \in \mathfrak{S}_m} [x_{ik_{\sigma(1)}}^{\pm}, [x_{ik_{\sigma(2)}}^{\pm}, \cdots, [x_{ik_{\sigma(m)}}^{\pm}, x_{jl}^{\pm}] \cdots] = 0 \qquad i \neq j \quad m = 1 - a_{ij}$$

for $k, l \in \mathbb{Z}$, where we set $x_{ik}^+ = e_{ik}$, $x_{ik}^- = f_{ik}$ and $[x, y]_+ = xy + yx$ for $x, y \in \mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$.

These relations are the same as in [KT] with $\overline{h} = 1$. The set $\{h_{ik}, x_{ik}^{\pm} | 1 \leqslant i \leqslant N-1, k \in \mathbb{Z}_{\geqslant 0}\}$ provides the Drinfel'd generators of $Y_{\overline{h}}(\mathfrak{sl}_N)$ [Dr3].

Let us set

$$E_i^{\pm}(u) = \frac{1}{\overline{h}} e_{i,i+1}^{\pm}(u_{\mp} + \tfrac{1}{2}\overline{h}i) \qquad F_i^{\pm}(u) = \frac{1}{\overline{h}} f_{i+1,i}^{\pm}(u_{\pm} + \tfrac{1}{2}\overline{h}i)$$

so that $E_i^{\pm}(u) = E_i^+(u) - E_i^-(u)$, $F_i^{\pm}(u) = F_i^+(u) - F_i^-(u)$. (See theorem 3.2 for the definition of $e_{i,i+1}^{\pm}(u)$ and $f_{i+1,i}^{\pm}(u)$.) Let us denote $\mathcal{D}Y$ for $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$ and $\mathcal{D}Y^{\pm}$ be the subalgebra of $\mathcal{D}Y$ generated by $e_{i,k}$, $f_{i,k}$, respectively. Set

$$\mathcal{N}^{\pm} = \sum_{i,k} x_{ik}^{\pm} \mathcal{D}Y^{\pm}.$$

We get the partial results of the coproduct of the currents $E_i^{\pm}(u)$, $F_i^{\pm}(u)$, $H_i^{\pm}(u)$, $K^{\pm}(u)$ which is sufficient for our purpose.

Lemma 3.6.

(i) $$\Delta(E_i^{\pm}(u)) \equiv E_i^{\pm}(u) \otimes 1 + H_i^{\pm}(u_{\mp}) \otimes E_i^{\pm}(u \mp \tfrac{1}{2}\overline{h}c_1)$$

(ii) $$\Delta(F_i^{\pm}(u)) \equiv 1 \otimes F_i^{\pm}(u) + F_i^{\pm}(u \pm \tfrac{1}{2}\overline{h}c_2) \otimes H_i^{\pm}(u_{\pm})$$

(iii) $$\Delta(H_i^{\pm}(u)) \equiv H_i^{\pm}(u \pm \tfrac{1}{4}\overline{h}c_2) \otimes H_i^{\pm}(u \mp \tfrac{1}{4}\overline{h}c_1)$$

$$\mathrm{mod}(\mathcal{N}^- \mathcal{D}Y \otimes \mathcal{D}Y \mathcal{N}^+) \cap (\mathcal{D}Y \mathcal{N}^- \otimes \mathcal{N}^+ \mathcal{D}Y)$$

(iv) $$\Delta(K^{\pm}(u)) = K^{\pm}(u \pm \tfrac{1}{4}\overline{h}c_2) \otimes K^{\pm}(u \mp \tfrac{1}{4}\overline{h}c_1).$$

The last formula follows from the fact that the equation $K^{\pm}(u) = q\text{-det}.T^{\pm}(u)$ holds. We remark that these formula for $Y_{\hbar}(\mathfrak{sl}_N)$ are obtained in [CP2]. The exact formulae in the case of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_2)$ are given in [IK]. For more information on the coproduct formulae, see appendix B.

3.4. Quantum current

Here we give a remark concerning definition 3.1 and introduce the so-called quantum current [RS].

Let $\overline{R}(u)$ be Yang's R-matrix normalized as (1) and $R(u) = f_N(u)\overline{R}(u)$ with some scalar function $f_N(u)$. We remark that even if we change the normalization of R-matrix in definition 3.1 to $R(u)$ defined here, the commutation relations given by corollary 3.4 never change except for the relations between $K^{\pm}(u)$. It changes as follows:

$$f(u_- - v_+)K^+(u)K^-(v) = K^-(v)K^+(u)f(u_+ - v_-)$$

where

$$f(u) = \left\{\prod_{i,j=1}^{N} f_N(u + (i-j)\overline{h})\right\}\prod_{k=1}^{N-1}\frac{u - k\overline{h}}{u + k\overline{h}}.$$

If we set

$$f_N(u) = \frac{\Gamma(u/N\overline{h})\Gamma(1 + u/N\overline{h})}{\Gamma(1/N + u/N\overline{h})\Gamma(1 - 1/N + u/N\overline{h})}$$

where $\Gamma(u)$ is the Euler's Gamma function, then $f(u) = 1$. In the rest of this subsection, we fix the function $f_N(u)$ as above. Let us define the quantum current $T(u)$ as

$$T(u) = T^+(u_-)T^-(u_+)^{-1}.$$

They satisfy the following commutation relations.

Lemma 3.7.

$$R(u - v)\overset{1}{T}(u)R_{21}(v - u - \overline{h}c)\overset{2}{T}(v) = \overset{2}{T}(v)R(u - v - \overline{h}c)\overset{1}{T}(u)R_{21}(v - u)$$

$$R(u_- - v_+)\overset{1}{T}(u)R_{21}(v_- - u_+)\overset{2}{T}{}^{\pm}(v_{\pm}) = \rho_N(u_{\mp} - v_{\pm})\overset{2}{T}{}^{\pm}(v_{\pm})\overset{1}{T}(u)$$

where $\rho_N(u) = f_N(u)f_N(-u)$.

Since $T(u)$ can be regarded as a $N \times N$ matrix, we can define the current $l(u)$ by

$$l(u) = \text{tr}.T(u).$$

At the critical level ($c = -N$), one can show that $l(u)$ commutes with $T^{\pm}(u)$ so that $l(u)$ provides the Yangian deformed Gelfand–Dickii algebra [FR].

Remark . Everything given in this section makes sense as a formal series in $\overline{h}$ except for this subsection. Since the function $f_N(u)$ chosen here cannot be regarded as a formal series in $\overline{h}$, the formulae given after the specific choice of $f_N(u)$ must be considered only as asymptotics.

4. Representation theory of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$

Unfortunately, we have no general theorem about the representation theory of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ at the moment due to the lack of triangular decomposition and the grading operator d. Nevertheless, we expect that the representation theory of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ can be established just as in the case of quantum affine algebra [CP, J].

In this section we present examples of both finite and infinite-dimensional representations of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$.

4.1. Finite-dimensional representations

At $c = 0$, the Heisenberg subalgebra becomes central in $\mathcal{D}Y_{\overline{h}}(\mathfrak{gl}_N)$. Hence by Schur's lemma, it is sufficient for investigating the irreducible finite-dimensional representations to consider the $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$ case. From corollary 3.5, we expect that most of the finite-dimensional $Y_{\overline{h}}(\mathfrak{sl}_N)$-module can be extended to $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$-module.

Let $\boldsymbol{d} = \{d_{i,k}\}_{1\leqslant i\leqslant N-1, k\in\mathbb{Z}}$ be a subset of $\mathcal{A}$. A $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$-module V is called the *pseudo-highest-weight module with pseudo-highest weight* $\boldsymbol{d}$ if there is an unique, up to scalar multiple, non-zero vector $v \in V$ such that V is generated by v and

$$h_{i,k}v = d_{i,k}v \qquad e_{i,k}v = 0 \qquad 1 \leqslant \forall\ i \leqslant N-1 \quad \forall\ k \in \mathbb{Z}.$$

Here we borrow this terminology from [CP]. Let us denote $V(\boldsymbol{d})$ for such V.

Conjecture 1. (i) Let V be an irreducible finite-dimensional $Y_{\overline{h}}(\mathfrak{sl}_N)$-module whose constant term of the Drinfel'd polynomials are invertible. Then V can be lift up to an irreducible finite-dimensional $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$-module.

(ii) The irreducible $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$-module $V(\boldsymbol{d})$ of pseudo-highest weight $\boldsymbol{d}$ is finite dimensional iff there exist monic polynomials $P_i(v) \in \mathcal{A}[v]$ $1 \leqslant i \leqslant N-1$ such that

$$1 - \overline{h}\sum_{k<0} d_{ik}v^{-k-1} = \frac{P_i(v + \frac{1}{2}(\alpha_i, \alpha_i)\overline{h})}{P_i(v)} = 1 + \overline{h}\sum_{k\geqslant 0} d_{ik}v^{-k-1}$$

in the sense that the left-hand side and the right-hand side are the Laurent expansion of the middle term about 0 and ∞, respectively.

The above monic polynomials P_i are called Drinfel'd polynomials. Next we show some examples which support this conjecture.

Example 4.1 (the $\mathfrak{sl}_2$ case). Here we omit writing the subscript 1 for simplicity. Let $W_m = \oplus_{j=0}^{m}\mathcal{A}w_j$ be the spin-$\frac{m}{2}$ representation of $\mathfrak{sl}_2$ and set

$$W_m(u) = W_m \otimes_{\mathcal{A}} \mathcal{A}((u^{-1}))$$

where u is thought to be either an indeterminate or an invertible element of $\mathcal{A}$. It is known by [CP1] that we can define the $Y_{\overline{h}}(\mathfrak{sl}_2)$-module structure on $W_m(u)$. It immediately follows that their action of $Y_{\overline{h}}(\mathfrak{sl}_2)$ can be extended to $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_2)$.

Lemma 4.1. The action $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_2)$ is given by

(i) $$e_k w_i = \left\{u + (\tfrac{1}{2}m - i + \tfrac{1}{2})\overline{h}\right\}^k (m - i + 1)w_{i-1}$$

(ii) $$f_k w_i = \left\{u + (\tfrac{1}{2}m - i - \tfrac{1}{2})\overline{h}\right\}^k (i + 1)w_{i+1}$$

(iii) $$h_k w_i = \Big[\left\{u + (\tfrac{1}{2}m - i - \tfrac{1}{2})\overline{h}\right\}^k (i+1)(m-i) - \left\{u + (\tfrac{1}{2}m - i + \tfrac{1}{2})\overline{h}\right\}^k i(m - i + 1)\Big] w_i$$

where we set $w_{-1} = w_{m+1} = 0$.

As a consequence, we obtain the following.

Corollary 4.2. (i) $W_m(u)$ is a pseudo-highest-weight module with pseudo-highest-weight $\boldsymbol{d} = \{d_k\}$ given by

$$d_k = m\left(u + \frac{m-1}{2}\overline{h}\right)^k.$$

(ii) The Drinfel'd polynomial P associated with $W_m(u)$ is given by

$$P(v) = \left\{v - u - \frac{m-1}{2}\hbar\right\}\left\{v - u - \frac{m-3}{2}\hbar\right\}\cdots\left\{v - u + \frac{m-1}{2}\hbar\right\}.$$

Example 4.2 (the $\mathfrak{sl}_N$ *case (vector representation)).* Let u be either an indeterminate or an invertible element of $\mathcal{A}$. Set

$$V_u = V \otimes_{\mathcal{A}} \mathcal{A}((u^{-1})) \qquad V = \oplus_{j=0}^{N-1} \mathcal{A} w_j.$$

We can extend the action of $Y_{\hbar}(\mathfrak{sl}_N)$ to $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ as follows (see example 2.1).

Lemma 4.3. The action of $\mathcal{D}Y(\mathfrak{sl}_N)$ is given by

(i) $$e_{i,k} w_i = \left(u - \frac{N-1-i}{2}\hbar\right)^k w_{i-1} \qquad e_{i,k} w_j = 0 \quad j \neq i$$

(ii) $$f_{i,k} w_{i-1} = \left(u - \frac{N-1-i}{2}\hbar\right)^k w_i \qquad f_{i,k} w_j = 0 \quad j \neq i-1$$

(iii) $$h_{i,k} w_{i-1} = \left(u - \frac{N-1-i}{2}\hbar\right)^k w_{i-1}$$

$$h_{i,k} w_i = -\left(u - \frac{N-1-i}{2}\hbar\right)^k w_i \qquad h_{i,k} w_j = 0 \quad j \neq i, i-1\,.$$

Hence we have the following.

Corollary 4.4. (i) V_u is a pseudo-highest-weight module with highest weight $\boldsymbol{d} = \{d_{ik}\}$ given by

$$d_{1k} = \left(u - \frac{N-2}{2}\hbar\right)^k \qquad d_{ik} = 0 \quad i \neq 1.$$

(ii) The Drinfel'd polynomials P_i associated to V_u are given by

$$P_1(v) = v - \left(u - \frac{N-2}{2}\hbar\right) \qquad P_i(v) = 1 \quad i \neq 1.$$

The next example is the generalization of the above example.

Example 4.3 (the $\mathfrak{sl}_N$ *case).* Let $\mathfrak{g}$ be a Lie algebra of type A_{N-1}. In example 2.2, we define irreducible finite-dimensional $Y_{\hbar}(\mathfrak{g})$-modules $V_a(m\Lambda_i)$ for $1 \leqslant i \leqslant N-1$. Here we give a sketch of proof that we can extend its action to $\mathcal{D}Y_{\hbar}(\mathfrak{g})$. To see this (i) Define $\mathcal{D}Y_{\hbar}(\mathfrak{g})$-module structure for $m = 1, \forall\, i$. (Calculate the action on each weight vector explicitly, then it turns out that the invertibilty in the conjecture is essential.)

(ii) Using the following embedding of $Y_{\hbar}(\mathfrak{g})$-module

$$V_a((m+1)\Lambda_i) \hookrightarrow V_{a-\frac{1}{2}\hbar}(m\Lambda_i) \otimes V_{a+\frac{m}{2}\hbar}(\Lambda_i)$$

prove that we can define $\mathcal{D}Y_{\hbar}(\mathfrak{g})$-module structure for $\forall\, m, \forall\, i$ by induction on m.

The details are left to the reader as an exercise.

Note that the Drinfel'd polynomials of $V_a(m\Lambda_i)$ are exactly the same as in example 2.2.

4.2. Bosonization of the level 1 module

Here we construct level 1 $\mathcal{D}Y_{\hbar}(\mathfrak{g})$-module and vertex operators for $\mathfrak{g} = \mathfrak{gl}_N, \mathfrak{sl}_N$ directly in terms of bosons.

Let $\mathfrak{h} = \oplus_{i=1}^{N} \mathbb{C}\varepsilon_i$ be a Cartan subalgebra of $\mathfrak{gl}_N$, $\overline{Q} = \oplus_{i=1}^{N-1} \mathbb{Z}\alpha_i$ ($\alpha_i = \varepsilon_i - \varepsilon_{i+1}$) be the root lattice of $\mathfrak{sl}_N$, $\overline{\Lambda}_i = \Lambda_i - \Lambda_0$ be the classical part of the ith fundamental weight and $(\cdot,\cdot)$ be the standard bilinear form defined by $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. Let us introduce bosons $\{a_{i,k} | 1 \leqslant i \leqslant N,\ k \in \mathbb{Z} \setminus \{0\}\}$ satisfying

$$[a_{i,k}, a_{j,l}] = k\delta_{i,j}\delta_{k+l,0}.$$

4.2.1. The $\mathfrak{gl}_N$ case. Set

$$\mathcal{F}_{i,s} := \mathcal{A}[a_{j,-k}(1 \leqslant j \leqslant N,\ k \in \mathbb{Z}_{>0})] \otimes \mathcal{A}[\overline{Q}]e^{\overline{\Lambda}_i + s(\sum_{j=1}^{N} \varepsilon_j)/N} \qquad (0 \leqslant i \leqslant N-1)$$

where s is a complex parameter and $\mathcal{A}[\overline{Q}]$ is the group algebra of $\overline{Q}$ over $\mathcal{A}$. On this space, we define the action of the operators $a_{j,k}, \partial_{\varepsilon_j}, e^{\varepsilon_j}$ $(1 \leqslant j \leqslant N)$ by

$$a_{j,k} \cdot f \otimes e^{\beta} = \begin{cases} a_{j,k} f \otimes e^{\beta} & k < 0 \\ [a_{j,k}, f] \otimes e^{\beta} & k > 0 \end{cases}$$

$$\partial_{\varepsilon_j} \cdot f \otimes e^{\beta} = (\varepsilon_j, \beta) f \otimes e^{\beta} \qquad \text{for} \quad f \otimes e^{\beta} \in \mathcal{F}_{i,s}$$

$$e^{\varepsilon_j} \cdot f \otimes e^{\beta} = f \otimes e^{\varepsilon_j + \beta}.$$

Theorem 4.5. The following assignment defines a $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$-module structure on $\mathcal{F}_{i,s}$.

$$k_j^+(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,k}}{k} \left\{\left(u + \frac{1}{2}\hbar\right)^{-k} - \left(u - \frac{1}{2}\hbar\right)^{-k}\right\}\right] \left(\frac{u - \frac{1}{2}\hbar}{u + \frac{1}{2}\hbar}\right)^{\partial_{\varepsilon_j}}$$

$$k_j^-(u) \mapsto \exp\left[\sum_{k>0, r<j} \frac{a_{r,-k}}{k} \left\{u^k - (u-\hbar)^k\right\} + \sum_{k>0, r>j} \frac{a_{r,-k}}{k} \left\{(u+\hbar)^k - u^k\right\}\right]$$

$$\frac{1}{\hbar} X_j^+(u) \mapsto \exp\left[-\sum_{k>0} \frac{a_{j,-k}}{k}\left(u - \frac{3}{4}\hbar\right)^k + \sum_{k>0} \frac{a_{j+1,-k}}{k}\left(u + \frac{1}{4}\hbar\right)^k\right]$$
$$\times \exp\left[\sum_{k>0} \frac{a_{j,k} - a_{j+1,k}}{k}\left(u + \frac{1}{4}\hbar\right)^{-k}\right] e^{\alpha_j} \left[(-1)^{j-1}\left(u + \frac{1}{4}\hbar\right)\right]^{\partial_{\alpha_j}}$$

$$\frac{1}{\hbar} X_j^-(u) \mapsto \exp\left[\sum_{k>0} \frac{a_{j,-k}}{k}\left(u - \frac{1}{4}\hbar\right)^k - \sum_{k>0} \frac{a_{j+1,-k}}{k}\left(u + \frac{3}{4}\hbar\right)^k\right]$$
$$\times \exp\left[\sum_{k>0} \frac{-a_{j,k} + a_{j+1,k}}{k}\left(u - \frac{1}{4}\hbar\right)^{-k}\right] e^{-\alpha_j} \left[(-1)^{j-1}\left(u - \frac{1}{4}\hbar\right)\right]^{-\partial_{\alpha_j}}$$

where we set $\partial_{\alpha_j} = \partial_{\varepsilon_j} - \partial_{\varepsilon_{j+1}}$.

Next we present the bosonization of type I and type II vertex operators. For this purpose, let us consider the evaluation module. Set

$$V_u = V \otimes_{\mathcal{A}} \mathcal{A}((u^{-1})) \qquad V = \oplus_{j=0}^{N-1} \mathcal{A} w_j.$$

We define the $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$-module structure on V_u as follows:

$$k^{\pm}_{i+1}(v)w_i = f^{\pm}(v-u)\frac{v-u+(\frac{1}{2}(N-3)-i)\hbar}{v-u+(\frac{1}{2}(N-1)-i)\hbar}w_i$$

$$k^{\pm}_j(v)w_i = f^{\pm}(v-u)w_i \quad \text{otherwise}$$

$$X^+_i(v)w_i = \hbar\delta\left(v-u+\left(\frac{N-1}{2}-i\right)\hbar\right)w_{i-1} \qquad X^+_j(v)w_i = 0 \quad \text{otherwise}$$

$$X^-_i(v)w_{i-1} = \hbar\delta\left(v-u+\left(\frac{N-1}{2}-i\right)\hbar\right)w_i \qquad X^-_j(v)w_i = 0 \quad \text{otherwise}$$

where we set

$$f^+(u) = 1 \qquad f^-(u) = \frac{u-\frac{1}{2}(N-1)\hbar}{u+\frac{1}{2}(N-3)\hbar}.$$

We remark that the restriction of the action of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ on the above V_u to that of $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ gives V_u in example 4.2 exactly.

Definition 4.1. The vertex operators are intertwiners of the following form:

(i) type I: $\quad \Phi^{(i,i+1)}(u) : \mathcal{F}_{i+1,s} \longrightarrow \mathcal{F}_{i,s-1} \otimes V_u$

(ii) type II: $\quad \Psi^{(i,i+1)}(u) : \mathcal{F}_{i+1,s} \longrightarrow V_u \otimes \mathcal{F}_{i,s-1}$.

Here the indices are considered modulo N.

Set

$$\Phi^{(i,i+1)}(u) = \sum_{j=0}^{N-1} \Phi^{(ii+1)}_j(u) \otimes w_j \qquad \Psi^{(i,i+1)}(u) = \sum_{j=0}^{N-1} w_j \otimes \Psi^{(ii+1)}_j(u)$$

We normalize them as

(i) $\quad \langle \Lambda_i, s-1|\Phi^{(i,i+1)}_i(u)|\Lambda_{i+1}, s\rangle = 1$

(ii) $\quad \langle \Lambda_i, s-1|\Psi^{(i,i+1)}_i(u)|\Lambda_{i+1}, s\rangle = 1$

where we set $|\Lambda_i, s\rangle = 1 \otimes e^{\overline{\Lambda}_i + s(\sum_{j=1}^{N}\varepsilon_j)/N}$. We mean by $\langle \Lambda_i, s-1|\Phi^{(i,i+1)}_i(u)|\Lambda_{i+1}, s\rangle$ the coefficient of $|\Lambda_i, s-1\rangle$ of the element $\Phi^{(i,i+1)}_i(u)|\Lambda_{i+1}, s\rangle$, and similarly for $\Psi^{(i,i+1)}_i(u)$. With the above normalization our vertex operators uniquely exist. By using lemma 3.6, we obtain the bosonization formula of these vertex operators as follows.

Theorem 4.6 (bosonization of vertex operators). For $0 \leqslant i \leqslant N-1$

$$\begin{aligned}\Phi^{(i,i+1)}_{N-1}(u) = \exp&\left[\sum_{k>0}\frac{a_{N,-k}}{k}\left(u+\left(\frac{N}{2}+\frac{1}{4}\right)\hbar\right)^k\right] \\ &\times\exp\left[\sum_{k>0;1\leqslant j<N}\frac{a_{j,k}}{k}\left(u-\left(\frac{N}{2}-\frac{1}{4}-j\right)\hbar\right)^{-k}\right] \\ &\times e^{-\varepsilon_N}\left[(-1)^{N-1}\left(u+\left(\frac{N}{2}-\frac{3}{4}\right)\hbar\right)\right]^{\partial_{\overline{\Lambda}_{N-1}}+(N-i-1)/N}(-1)^{\frac{1}{2}(N-i-1)(N+i-2)}\end{aligned}$$

$$\Phi^{(i,i+1)}_{k-1}(u) = [\Phi^{(i,i+1)}_{k}(u),\, f_{k,0}]$$

$$\Psi^{(i,i+1)}_{0}(u) = \exp\left[\sum_{k>0}\frac{a_{1,-k}}{k}\left(u-\left(\frac{N}{2}-\frac{3}{4}\right)\overline{h}\right)^{k}\right]$$

$$\times\exp\left[\sum_{k>0;1<j\leqslant N}\frac{a_{j,k}}{k}\left(u-\left(\frac{N}{2}+\frac{1}{4}-j\right)\overline{h}\right)^{-k}\right]$$

$$\times e^{-\varepsilon_1}\left[-\left(u-\left(\frac{N}{2}-\frac{7}{4}\right)\overline{h}\right)\right]^{-\partial_{\overline{\Lambda}_1}+(N-i-1)/N}(-1)^{\frac{1}{2}i(i+1)}$$

$$\Psi^{(i,i+1)}_{k}(u) = [\Psi^{(i,i+1)}_{k-1}(u), e_{k,0}].$$

4.2.2. The $\mathfrak{sl}_N$ case. Here we keep the same notation as in $\mathfrak{gl}_N$ case unless otherwise stated. Set

$$\mathcal{F}_i := \mathcal{A}[a_{j,-k}(1\leqslant j\leqslant N-1,\ k\in\mathbb{Z}_{>0})]\otimes\mathcal{A}[\overline{Q}]e^{\overline{\Lambda}_i}\qquad (0\leqslant i\leqslant N-1).$$

As in the previous subsection, we define the action of the operators $a_{j,k}$, ∂_{α_j}, e^{α_j} $(1\leqslant j\leqslant N-1)$ on $\mathcal{F}_i$.

Theorem 4.7. The following assignment defines a $\mathcal{D}Y_{\overline{h}}(\mathfrak{sl}_N)$-module structure on $\mathcal{F}_i$.

$$H_j^{+}(u)\mapsto\exp\left[-\sum_{k>0}\frac{a_{j,k}}{k}\left\{\left(u+\frac{1}{2}\overline{h}\right)^{-k}-\left(u-\frac{1}{2}\overline{h}\right)^{-k}\right\}\right]\left(\frac{u-\frac{1}{2}\overline{h}}{u+\frac{1}{2}\overline{h}}\right)^{-\partial_{\alpha_j}}$$

$$H_j^{-}(u)\mapsto\exp\left[-\sum_{k>0}\frac{a_{j,-k}}{k}\left\{(u+\overline{h})^{k}-(u-\overline{h})^{k}\right\}\right.$$

$$\left.+\sum_{k>0}\frac{a_{j+1,-k}+a_{j-1,-k}}{k}\left\{\left(u+\frac{1}{2}\overline{h}\right)^{k}-\left(u-\frac{1}{2}\overline{h}\right)^{k}\right\}\right]$$

$$E_j(u)\mapsto\exp\left[\sum_{k>0}\frac{a_{j,-k}}{k}\left\{\left(u+\frac{1}{4}\overline{h}\right)^{k}+\left(u-\frac{3}{4}\overline{h}\right)^{k}\right\}\right.$$

$$\left.-\sum_{k>0}\frac{a_{j+1,-k}+a_{j-1,-k}}{k}\left(u-\frac{1}{4}\overline{h}\right)^{k}\right]\exp\left[-\sum_{k>0}\frac{a_{j,k}}{k}\left(u+\frac{1}{4}\overline{h}\right)^{-k}\right]$$

$$\times e^{\alpha_j}\left[(-1)^{j-1}\left(u+\frac{1}{4}\overline{h}\right)\right]^{\partial_{\alpha_j}}$$

$$F_j(u)\mapsto\exp\left[-\sum_{k>0}\frac{a_{j,-k}}{k}\left\{\left(u+\frac{3}{4}\overline{h}\right)^{k}+\left(u-\frac{1}{4}\overline{h}\right)^{k}\right\}\right.$$

$$\left.+\sum_{k>0}\frac{a_{j+1,-k}+a_{j-1,-k}}{k}\left(u+\frac{1}{4}\overline{h}\right)^{k}\right]\exp\left[\sum_{k>0}\frac{a_{j,k}}{k}\left(u-\frac{1}{4}\overline{h}\right)^{-k}\right]$$

$$\times e^{-\alpha_j}\left[(-1)^{j-1}\left(u-\frac{1}{4}\overline{h}\right)\right]^{-\partial_{\alpha_j}}.$$

Before investigating the vertex operators, we shall give some remarks here. Every field in theorems 4.5–4.7 make sense as a formal series in $\hbar$ if we use the binomial expansion

$$(u+a\hbar)^k = \sum_{j\geqslant 0} \binom{k}{j} (a\hbar)^j u^{k-j} \qquad a \in \mathcal{A}, k \in \mathbb{Z}.$$

Now one can prove these theorems by some routine calculations. Notice that because of the artificial choice of the action of the Heisenberg subalgebra, the bosonization of the vertex operators in the case of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$ has such nice expression. For the $\mathcal{D}Y_{\hbar}(\mathfrak{sl}_N)$ case, as we will see soon, we have some subtle problem to bosonize the vertex operators.

To introduce the vertex operators of type I and type II, let us fix the evaluation module V_u given in example 4.2.

Definition 4.2. The vertex operators are intertwiners of the following form:

(i) type I: $\Phi^{(i,i+1)}(u) : \mathcal{F}_{i+1} \longrightarrow \mathcal{F}_i \otimes V_u$

(ii) type II: $\Psi^{(i,i+1)}(u) : \mathcal{F}_{i+1} \longrightarrow V_u \otimes \mathcal{F}_i$.

Here the indices are considered modulo N.

Set

$$\Phi^{(i,i+1)}(u) = \sum_{j=0}^{N-1} \Phi_j^{(ii+1)}(u) \otimes w_j \qquad \Psi^{(i,i+1)}(u) = \sum_{j=0}^{N-1} w_j \otimes \Psi_j^{(ii+1)}(u)$$

We normalize them as

(i) $\langle \Lambda_i | \Phi_i^{(i,i+1)}(u) | \Lambda_{i+1} \rangle = 1$

(ii) $\langle \Lambda_i | \Psi_i^{(i,i+1)}(u) | \Lambda_{i+1} \rangle = 1$

Just as in the case of $\mathcal{D}Y_{\hbar}(\mathfrak{gl}_N)$, our vertex operators uniquely exist and the bosonization formulae are as follows.

Theorem 4.8 (bosonization of vertex operators). For $0 \leqslant i \leqslant N-1$

$$\Phi_{N-1}^{(i,i+1)}(u) = \lim_{n\to\infty} \Phi_{N-1}^{(i,i+1)}(u)_n$$

$$\begin{aligned}
\Phi_{N-1}^{(i,i+1)}(u)_n &= \exp\left[\sum_{k>0} \frac{a_{N-1,-k}}{k}\left(u+\frac{3}{4}\hbar\right)^k\right] \exp\left[-\sum_{k>0;1\leqslant j<N} \frac{a_{j,k}}{2k} f_{j,k}^{(\mathrm{I})}\left(u-\tfrac{1}{4}\hbar\right)_n\right] \\
&\quad \times e^{\overline{\Lambda}_{N-1}} \left\{\prod_{j=1}^{N-1} \left[g_j^{(\mathrm{I})}\left(u-\tfrac{1}{4}\hbar\right)_n\right]^{\partial_{\alpha_j}}\right\} \left[(-1)^{(N-1)/2}(N\hbar)^{-(N-2)/2}\right]^{\partial_{\overline{\Lambda}_{N-1}}-(i+1)/N} \\
&\quad \times g_{i+1}^{(\mathrm{I})}\left(u-\tfrac{1}{4}\hbar\right)_n^{-1} \left[u-\left(\frac{N-i}{2}-\frac{3}{4}\right)\hbar\right] (-1)^{\frac{1}{2}(N-i)(N+i-1)}
\end{aligned}$$

$$\Phi_{k-1}^{(i,i+1)}(u) = [\Phi_k^{(i,i+1)}(u), f_{k,0}]$$

$$\Psi_0^{(i,i+1)}(u) = \lim_{n\to\infty} \Psi_0^{(i,i+1)}(u)_n$$

$$\Psi_0^{(i,i+1)}(u)_n = \exp\left[-\sum_{k>0} \frac{a_{1,-k}}{k}\left(u-\left(\frac{N}{2}-\frac{1}{4}\right)\hbar\right)^k\right] \exp\left[\sum_{k>0;1\leqslant j<N} \frac{a_{j,k}}{2k} f_{N-j,k}^{(\mathrm{II})}\left(u+\tfrac{1}{4}\hbar\right)_n\right]$$

$$\times e^{-\overline{\Lambda}_1}\left\{\prod_{j=1}^{N-1}\left[g^{(\mathrm{II})}_{N-j}\left(u+\tfrac{1}{4}\overline{h}\right)_n\right]^{-\partial_{\alpha_j}}\right\}\left[(-1)^{-\frac{1}{2}}(N\overline{h})^{-(N-2)/2}\right]^{-\partial_{\overline{\Lambda}_1}+(N-i-1)/N}$$

$$\times g^{(\mathrm{II})}_{N-i-1}\left(u+\tfrac{1}{4}\overline{h}\right)_n(-1)^{\frac{1}{2}i(i+1)}$$

$\Psi_k^{(i,i+1)}(u)=[\Psi_{k-1}^{(i,i+1)}(u),e_{k,0}]$.

Here the functions $f^*_{j,k}(u)_n$, $g^*_j(u)_n$ $(*=(\mathrm{I}),(\mathrm{II}))$ are defined as follows:

$$f^*_{j,k}(u)_n=\sum_{l=0}^{j-1}f^*_k\left(u+\tfrac{1}{2}(j-1-2l)\overline{h}\right)_n\qquad 1\leqslant j<N$$

$$g^*_j(u)_n=\begin{cases}1 & j=0\\ \left[\prod_{l=0}^{j-1}g^*\left(u+\tfrac{1}{2}(j-1-2l)\overline{h}\right)_n\right]^{\frac{1}{2}} & j>0\end{cases}$$

$$f^{(\mathrm{I})}_k(u)_n=\left(u-\frac{N-2}{2}\overline{h}\right)^{-k}+\sum_{l=0}^{n-1}\left\{\left(u+\frac{N}{2}\overline{h}+N\overline{h}l\right)^{-k}-\left(u+\frac{N}{2}\overline{h}+(Nl+1)\overline{h}\right)^{-k}\right.$$

$$\left.+\left(u-\frac{3N}{2}\overline{h}-(Nl-1)\overline{h}\right)^{-k}-\left(u-\frac{N}{2}\overline{h}-N\overline{h}l\right)^{-k}\right\}$$

$$f^{(\mathrm{II})}_k(u)_n=u^{-k}+\sum_{l=0}^{n-1}\left\{(u+N\overline{h}+N\overline{h}l)^{-k}-(u+\overline{h}+N\overline{h}l)^{-k}\right.$$

$$\left.+(u-(N-1)\overline{h}-N\overline{h}l)^{-k}-(u-N\overline{h}-N\overline{h}l)^{-k}\right\}$$

$$g^{(\mathrm{I})}(u)_n=\left(u-\frac{N-2}{2}\overline{h}\right)e^{(N-2)\gamma/N}\frac{(u+\frac{1}{2}N\overline{h})(u-\frac{3}{2}N\overline{h}+\overline{h})}{(u-\frac{1}{2}N\overline{h})(u+\frac{1}{2}N\overline{h}+\overline{h})}$$

$$\times\prod_{l=1}^{n-1}\left[\frac{(u+\frac{1}{2}N\overline{h}+N\overline{h}l)(u-\frac{3}{2}N\overline{h}-(Nl-1)\overline{h})}{(u-\frac{1}{2}N\overline{h}-N\overline{h}l)(u+\frac{1}{2}N\overline{h}+(Nl+1)\overline{h})}\right]e^{(N-2)/Nl}$$

$$g^{(\mathrm{II})}(u)_n=ue^{(N-2)\gamma/N}\frac{(u+N\overline{h})(u-(N-1)\overline{h})}{(u+\overline{h})(u-N\overline{h})}$$

$$\times\prod_{l=1}^{n-1}\left[\frac{(u+N\overline{h}+N\overline{h}l)(u-(N-1)\overline{h}-N\overline{h}l)}{(u+\overline{h}+N\overline{h}l)(u-N\overline{h}-N\overline{h}l)}\right]e^{(N-2)/Nl}$$

where γ is the Euler constant defined by

$$\gamma=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right).$$

We remark that the following formulae hold:

(i) $$\exp\left[-\sum_{k>0}\frac{1}{2k}f^*_k(u)_n v^k\right]=\left[\frac{g^*(u-v)_n}{g^*(u)_n}\right]^{\frac{1}{2}}\qquad\text{for }\ *=(\mathrm{I}),(\mathrm{II})$$

(ii)
$$\lim_{n\to\infty} g^*(u)_n = \begin{cases} \left(u - \dfrac{N-2}{2}\overline{h}\right) \dfrac{\Gamma(\frac{1}{2} - u/N\overline{h})\Gamma(\frac{1}{2} + (u+\overline{h})/N\overline{h})}{\Gamma(\frac{1}{2} + u/N\overline{h})\Gamma(\frac{3}{2} - (u+\overline{h})/N\overline{h})} \\ \qquad \text{for } * = (\mathrm{I}) \\ u \dfrac{\Gamma((u+\overline{h})/N\overline{h})\Gamma(1 - u/N\overline{h})}{\Gamma(1 - (u+\overline{h})/N\overline{h}))\Gamma(1 + u/N\overline{h})} \\ \qquad \text{for } * = (\mathrm{II}). \end{cases}$$

The second formula can be proved by using the famous Weierstrass formula for the Gamma function

$$\frac{1}{\Gamma(u)} = u e^{\gamma u} \prod_{n=1}^{\infty} \left(1 + \frac{u}{n}\right) e^{-u/n}.$$

Remark . For each $n \in \mathbb{Z}_{>0}$, the fields $\Phi_{N-1}^{(i,i+1)}(u)_n$ and $\Psi_0^{(i,i+1)}(u)_n$ make sense as formal series in $\overline{h}$. But after taking the limit $n \to \infty$, they cannot expand with respect to $\overline{h}$. They have to be regarded as, for example, meromorphic functions. Such a feature has never appeared in the quantum affine case [K].

Here we give a sketch of a proof of theorem 4.8 for the type I vertex operator, considering the $N = 2$ case for simplicity. We also give some comments on how to prove the general case.

We define the normal ordering $: \cdot :$ of the fields by regarding $a_{j,k}(k < 0), e^{\alpha_j}(1 \leqslant j \leqslant N-1)$ as creation operators and $a_{j,k}(k > 0), \partial_{\alpha_j}(1 \leqslant j \leqslant N-1)$ as annihilation operators. After some calculation, we obtain the following operator product expansion (OPE):

$$\Phi_1^{(i,i+1)}(u)_n H_1^-(v) = \left[\left(\frac{u - v + \frac{3}{4}\overline{h}}{u - v - \frac{1}{4}\overline{h}}\right)^2 \frac{(u - v + (2n - \frac{1}{4})\overline{h})(u - v - (2n + \frac{1}{4})\overline{h})}{(u - v + (2n + \frac{3}{4})\overline{h})(u - v - (2n + \frac{5}{4})\overline{h})}\right]^{\frac{1}{2}}$$
$$\times : \Phi_1^{(i,i+1)}(u)_n H_1^-(v) :$$

$$\Phi_1^{(i,i+1)}(u)_n E_1(v) = (-1)^{\frac{1}{2}} \left[(u - v + \tfrac{1}{2}\overline{h})^2 \frac{u - v - (2n + \frac{1}{2})\overline{h}}{u - v + (2n + \frac{1}{2})\overline{h}}\right]^{\frac{1}{2}} : \Phi_1^{(i,i+1)}(u)_n E_1(v) : .$$

Taking the limit $n \to \infty$, fixing the branch, we get

$$\Phi_1^{(i,i+1)}(u) H_1^-(v) = \frac{u - v + \frac{3}{4}\overline{h}}{u - v - \frac{1}{4}\overline{h}} : \Phi_1^{(i,i+1)}(u) H_1^-(v) :$$

$$\Phi_1^{(i,i+1)}(u) E_1(v) = -(u - v + \tfrac{1}{2}\overline{h}) : \Phi_1^{(i,i+1)}(u) E_1(v) : .$$

These are precisely the expected OPE from the intertwining property. The other OPEs can be obtained easily and we omit them here. The normalization condition can also be checked in a similar manner.

Next to prove the general case, first simplify the OPE, as above, to see the phase factor and then calculate the limit using the infinite product form of the Gamma function. In this way we can prove that our formulae give the desired OPE and the normalization.

5. Discussion

In this paper, we have constructed the Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ with a central extension for $\mathfrak{g} = \mathfrak{gl}_N.\mathfrak{sl}_N$. We also presented Drinfel'd generators which are defined in [Dr3]. Using these generators, we studied both finite- and infinite-dimensional representations. We presented a conjecture for the irreducible finite-dimensional representations and gave some examples to check the validity of them. The bosonization of the level 1 modules and the vertex operators were also given.

It seems that the Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ for another type of simple finite-dimensional Lie algebra $\mathfrak{g}$ can be defined by corollary 3.4 without $K^{\pm}(u)$ where $A = (a_{ij})$ is now the corresponding Cartan matrix. Suppose for a moment that this is true. Then the rest of section 3 also holds without any change. In particular, when $\mathfrak{g}$ is a simply laced algebra, we can generalize theorem 4.7 by a simple modification whose quantum affine version is treated in [FJ]. There are several other problems which we have already mentioned in our previous paper [IK]. The relation between the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ and the Yangian double $\mathcal{D}Y_{\hbar}(\mathfrak{g})$ is quite mysterious.

For physical applications, it is important to investigate the infinite-dimensional representation theory of $\mathcal{D}Y_{\hbar}(\mathfrak{g})$. In this paper, we give the bosonization of the level 1 module $\mathcal{F}_i$ and the vertex operators among them. As we have seen in theorem 4.8, the Fourier coefficients of the vertex operators loose sense unlike to the quantum affine case [JM, K]. This means that we have to consider not the Fourier components but the currents themselves. Namely we have to consider a new class of the algebra and their representation theory to investigate further. It is also interesting to see the connection between the formulae in [Lu] and ours.

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Appendix A. Review of quantum groups

In this appendix, we recall some facts about universal $\mathcal{R}$ and L-operator.

A.1. Universal $\mathcal{R}$

Let $\mathcal{R}$ be the universal R-matrix [Dr1] for $U_q(\hat{\mathfrak{sl}}_N)$. For the definition and the properties of universal R-matrix, see [Dr1, J].

We slightly modify $\mathcal{R}$ to define L-operators. Define

$$\mathcal{R}'^{+} = q^{-\frac{1}{2}(c\otimes d+d\otimes c)}\sigma(\mathcal{R}^{-1})q^{-\frac{1}{2}(c\otimes d+d\otimes c)}$$

$$\mathcal{R}'^{-} = q^{\frac{1}{2}(c\otimes d+d\otimes c)}\mathcal{R}q^{\frac{1}{2}(c\otimes d+d\otimes c)}$$

$$\mathcal{R}'^{\pm}(z) = (z^{d} \otimes \mathrm{id})\mathcal{R}'^{\pm}(z^{-d} \otimes \mathrm{id}).$$

Here σ stands for the flip of tensor components $\sigma(a\otimes b) = b\otimes a$. We remark that $\mathcal{R}'^{\pm}(z)$ are formal power series in $z^{\mp 1}$. The properties of universal R-matrix can be readily translated in terms of $\mathcal{R}'^{\pm}$. For $x \in U_q(\hat{\mathfrak{sl}}_N)$, we write $\Delta(x) = x_{(1)} \otimes x_{(2)}$. Then

$$\mathcal{R}'^{\pm}(z)\left(\mathrm{Ad}(z^d q^{\pm\frac{1}{2}c_2 d})x_{(1)} \otimes \mathrm{Ad}(q^{\pm\frac{1}{2}c_1 d})x_{(2)}\right)$$

$$= \left(\mathrm{Ad}(z^d q^{\mp\frac{1}{2}c_2 d})x_{(2)} \otimes \mathrm{Ad}(q^{\mp\frac{1}{2}c_1 d})x_{(1)}\right)\mathcal{R}'^{\pm}(z).$$

Here $c_1 = c \otimes 1$ and $c_2 = 1 \otimes c$ as in section 3.

The Yang–Baxter equation takes the form

$$\mathcal{R}'^{\pm}_{12}(z/w)\mathcal{R}'^{\pm}_{13}(zq^{\pm c_2})\mathcal{R}'^{\pm}_{23}(w) = \mathcal{R}'^{\pm}_{23}(w)\mathcal{R}'^{\pm}_{13}(zq^{\mp c_2})\mathcal{R}'^{\pm}_{12}(z/w)$$

$$\mathcal{R}'^{+}_{12}(z/wq^{-c_3})\mathcal{R}'^{+}_{13}(z)\mathcal{R}'^{-}_{23}(w) = \mathcal{R}'^{-}_{23}(w)\mathcal{R}'^{+}_{13}(z)\mathcal{R}'^{+}_{12}(z/wq^{c_3}).$$

For completeness we give the transformation properties of $\mathcal{R}'^{\pm}$ under the coproduct Δ, the counit ε and the antipode S.

$$(\Delta \otimes \mathrm{id})\,\mathcal{R}'^{\pm}(z) = \mathcal{R}'^{\pm}_{13}(zq^{\pm\frac{1}{2}c_2})\mathcal{R}'^{\pm}_{23}(zq^{\mp\frac{1}{2}c_1})$$

$$(\mathrm{id} \otimes \Delta)\,\mathcal{R}'^{\pm}(z) = \mathcal{R}'^{\pm}_{13}(zq^{\mp\frac{1}{2}c_2})\mathcal{R}'^{\pm}_{12}(zq^{\pm\frac{1}{2}c_3})$$

$$(\varepsilon \otimes \mathrm{id})\mathcal{R}'^{\pm}(z) = (\mathrm{id} \otimes \varepsilon)\mathcal{R}'^{\pm}(z) = 1$$

$$(S \otimes \mathrm{id})\mathcal{R}'^{\pm}(z) = (\mathrm{id} \otimes S^{-1})\mathcal{R}'^{\pm}(z) = \mathcal{R}'^{\pm}(z)^{-1}.$$

A.2. L-operators

Let now $\pi_V : U_q(\hat{\mathfrak{sl}}_N)' \to \mathrm{End}(V)$ be a finite-dimensional representation, where $U_q(\hat{\mathfrak{sl}}_N)'$ signifies the subalgebra of $U_q(\hat{\mathfrak{sl}}_N)$ with q^d being dropped. The evaluation representation π_{V_z} associated with V is defined by

$$\pi_{V_z}(x) = \pi_V\left(z^d x z^{-d}\right) \qquad \forall\ x \in U_q(\hat{\mathfrak{sl}}_N)'.$$

Introduce the L-operators

$$L^{\pm}(z) = L^{\pm}_V(z) = \left(\pi_{V_z} \otimes \mathrm{id}\right)\mathcal{R}'^{\pm}.$$

Taking the image of the Yang–Baxter equation for $\mathcal{R}'^{\pm}$ in $\mathrm{End}(V_z) \otimes \mathrm{End}(V_w) \otimes \mathrm{id}$, we find the following RLL relations:

$$R^{\pm}_{12}(z/w)\overset{1}{L}{}^{\pm}(z)\overset{2}{L}{}^{\pm}(w) = \overset{2}{L}{}^{\pm}(w)\overset{1}{L}{}^{\pm}(z)R^{\pm}_{12}(z/w)$$

$$R^{+}_{12}(q^{-c}z/w)\overset{1}{L}{}^{+}(z)\overset{2}{L}{}^{-}(w) = \overset{2}{L}{}^{-}(w)\overset{1}{L}{}^{+}(z)R^{+}_{12}(q^{c}z/w)$$

where we set

$$R^{\pm}(z/w) = \left(\pi_{V_z} \otimes \pi_{V_w}\right)\mathcal{R}'^{\pm}.$$

Introducing the matrix units E_{ij} let us define the entries $L^{\pm}_{ij}(z)$ by

$$L^{\pm}(z) = \sum E_{ij} \otimes L^{\pm}_{ij}(z).$$

In these terms, the Hopf algebra structure reads as follows:

$$\Delta\left(L^{\pm}_{ij}(z)\right) = \sum_k L^{\pm}_{kj}(q^{\pm\frac{1}{2}c_2}z) \otimes L^{\pm}_{ik}(q^{\mp\frac{1}{2}c_1}z)$$

$$\varepsilon\left(L^{\pm}_{ij}(z)\right) = \delta_{ij}$$

$$S\left({}^tL^{\pm}(z)\right) = \left({}^tL^{\pm}(z)\right)^{-1}$$

$$S^{-1}\left(L^{\pm}(z)\right) = \left(L^{\pm}(z)\right)^{-1}.$$

In the last two lines we set

$$S\left({}^tL^{\pm}(z)\right) = \sum E_{ji} \otimes S\left(L^{\pm}_{ij}(z)\right)$$

$$S^{-1}\left(L^{\pm}(z)\right) = \sum E_{ij} \otimes S^{-1}\left(L^{\pm}_{ij}(z)\right).$$

Let $U^{\pm}$ be Hopf subalgebras of $U_q(\hat{\mathfrak{sl}}_N)$ generated by $q^{\pm\frac{1}{2}c}$ and the Fourier components of $L^{\pm}(z)$. The subalgebra U^- is the dual Hopf algebra of U^+ with opposite comultiplication and the Hopf pairing between $U^{\pm}$ has the explicit description as follows:

$$\langle L^+(z), L^-(w)\rangle = \sum \langle L^+_{ij}(z), L^-_{kl}(w)\rangle E_{ij} \otimes E_{kl} = R^+(z/w).$$

We remark that all of these formulae motivate our choice of $T^{\pm}(u)$-matrix.

Appendix B. Several formulae for T-matrices

In this appendix, we collect some formulae which seem well known to the specialists [Ta]. Here we denote $T(u)$ for $T^{\pm}(u)$ for simplicity.

B.1. The quantum determinant of the T-matrix

In this subappendix, we give a brief review on quantum determinant for convenience. See [MNO, KS] for further information.

B.1.1. Quantum minor. Let V be a rank N $\mathcal{A}$-free module and $\mathcal{P} \in \mathrm{End}(V \otimes V)$ be a permutation operator $\mathcal{P}v \otimes w = w \otimes v$ $(v, w \in V)$. Let us fix the normalization of Yang's R-matrix as

$$R(u) = I + \frac{\overline{h}}{u}\mathcal{P} \in \mathrm{End}(V \otimes V).$$

Recall that $T(u)$ enjoy the following commutation relations:

$$R(u-v)\overset{1}{T}(u)\overset{2}{T}(v) = \overset{2}{T}(v)\overset{1}{T}(u)R(u-v).$$

Suppose the comultiplication of $T(u)$ is given by

$$\Delta(T(u)) = T(u)\dot{\otimes}T(u) \qquad \text{or equivalently} \qquad \Delta(t_{ij}(u)) = \sum_{k=1}^{N} t_{ik}(u) \otimes t_{kj}(u).$$

For simplicity, set $R_{i,j} = R_{i,j}(u_i - u_j)$ and

$$R(u_1, u_2, \cdots, u_p) = (R_{p-1,p})(R_{p-2,p}R_{p-2,p-1})\cdots(R_{1,p}R_{1,p-1}\cdots R_{1,2})$$

where the meaning of the lower indices are the same as in section 3.

Lemma B.1.

$$R(u_1, u_2, \cdots, u_p)\overset{1}{T}(u_1)\overset{2}{T}(u_2)\cdots\overset{p}{T}(u_p) = \overset{p}{T}(u_p)\cdots\overset{2}{T}(u_2)\overset{1}{T}(u_1)R(u_1, u_2, \cdots, u_p).$$

Let $\mathcal{A}[\mathfrak{S}_p]$ be the group algebra of the pth symmetric group over $\mathcal{A}$ which naturally acts on $V^{\otimes p}$ and set

$$a_p = \sum_{\sigma \in \mathfrak{S}_p} (\mathrm{sgn}\,\sigma)\sigma \in \mathcal{A}[\mathfrak{S}_p] \qquad A_p = \frac{1}{p!}a_p.$$

Lemma B.2 ([MNO]). For $u_i - u_{i+1} = -\overline{h}$, $1 \leqslant \forall\, i < p$

$$R(u_1, u_2, \cdots u_p) = a_p.$$

One can prove this lemma by induction on p. Combining these two lemmas, we obtain the following.

Lemma B.3.

$$A_p \overset{1}{T}\left(u-\frac{p-1}{2}\hbar\right)\overset{2}{T}\left(u-\frac{p-3}{2}\hbar\right)\cdots\overset{p}{T}\left(u+\frac{p-1}{2}\hbar\right)$$
$$=\overset{p}{T}\left(u+\frac{p-1}{2}\hbar\right)\cdots\overset{2}{T}\left(u-\frac{p-3}{2}\hbar\right)\overset{1}{T}\left(u-\frac{p-1}{2}\hbar\right)A_p.$$

Set $p=N$ in the above lemma. Since the Nth exterior power $\bigwedge^N V$ is of rank 1 and A_N stabilizes $\bigwedge^N V$, the left hand side of the above equation is (scalar) $\times\ A_N$.

Definition B.1 (quantum determinant).

$$q\text{-det}\,.T(u)A_N=\overset{N}{T}\left(u+\frac{N-1}{2}\hbar\right)\cdots\overset{2}{T}\left(u-\frac{N-3}{2}\hbar\right)\overset{1}{T}\left(u-\frac{N-1}{2}\hbar\right)A_N.$$

Explicitly, we have

Proposition B.4.

$$q\text{-det}\,T(u)=\sum_{\sigma\in\mathfrak{S}_N}(\operatorname{sgn}\sigma)t_{\sigma(1),1}\left(u-\frac{N-1}{2}\hbar\right)t_{\sigma(2),2}\left(u-\frac{N-3}{2}\hbar\right)$$
$$\cdots t_{\sigma(N),N}\left(u+\frac{N-1}{2}\hbar\right)$$
$$=\sum_{\sigma\in\mathfrak{S}_N}(\operatorname{sgn}\sigma)t_{1,\sigma(1)}\left(u+\frac{N-1}{2}\hbar\right)t_{2,\sigma(2)}\left(u+\frac{N-3}{2}\hbar\right)$$
$$\cdots t_{N,\sigma(N)}\left(u-\frac{N-1}{2}\hbar\right).$$

Next we explain some facts about quantum minors of the T-matrix. For two index subsets $I,J\subset\{1,2\cdots N\}$ with $\#I=\#J=p$, $1\leqslant p\leqslant N$ (the cardinality), set

$$T_{IJ}(u)=(t_{ij}(u))_{i\in I,j\in J}.$$

By the definition of $T(u)$, we obtain the following commutation relations:

$$R_p(u-v)\overset{1}{T}_{IJ}(u)\overset{2}{T}_{IJ}(v)=\overset{2}{T}_{IJ}(v)\overset{1}{T}_{IJ}(u)R_p(u-v)$$

$$R_p(u)=I+\frac{\hbar}{u}\mathcal{P}\in\operatorname{End}(V_p\otimes V_p)$$

where V_p is a rank-p $\mathcal{A}$-free module. Thus by an argument similar to that above, we get the explicit expression of quantum minor q-det $T_{IJ}(u)$ as follows. Set

$$I=\{i_1,i_2,\cdots,i_p\}\qquad J=\{j_1,j_2,\cdots,j_p\}.$$

Lemma B.5.

$$q\text{-det}\, T_{IJ}(u) = \sum_{\sigma\in\mathfrak{S}_p} (\operatorname{sgn}\sigma) t_{i_{\sigma(1)},j_1}\left(u - \frac{p-1}{2}\overline{h}\right) t_{i_{\sigma(2)},j_2}\left(u - \frac{p-3}{2}\overline{h}\right)$$

$$\cdots t_{i_{\sigma(N)},j_N}\left(u + \frac{p-1}{2}\overline{h}\right)$$

$$= \sum_{\sigma\in\mathfrak{S}_p} (\operatorname{sgn}\sigma) t_{i_1,j_{\sigma(1)}}\left(u + \frac{p-1}{2}\overline{h}\right) t_{i_2,j_{\sigma(2)}}\left(u + \frac{p-3}{2}\overline{h}\right)$$

$$\cdots t_{i_N,j_{\sigma(N)}}\left(u - \frac{p-1}{2}\overline{h}\right).$$

The following corollary is the immediate consequence of the above expression.

Corollary B.6. For each $\sigma \in \mathfrak{S}_p$

$$q\text{-det}\, T_{I^\sigma J}(u) = q\text{-det}\, T_{IJ^\sigma}(u) = (\operatorname{sgn}\sigma)\, q\text{-det}\, T_{IJ}(u)$$

where we set $I^\sigma = \{i_{\sigma^{-1}(1)}, i_{\sigma^{-1}(2)}, \cdots, i_{\sigma^{-1}(p)}\}$ and similarly for J^σ.

Using this corollary, one can calculate the coproduct of quantum minors as follows.

Corollary B.7.

$$\Delta(q\text{-det}\, T_{IJ}(u)) = \sum_K q\text{-det}\, T_{IK}(u) \otimes q\text{-det}\, T_{KJ}(u)$$

where the summation runs over all of the ordered subset $K = \{k_1, k_2, \cdots, k_p\} \subset S$ satisfying $1 \leqslant k_1 < k_2 < \cdots < k_p \leqslant N$.

B.1.2. Laplace expansion of the T-matrix. Let $\{e_i\}_{1\leqslant i\leqslant N}$ be an $\mathcal{A}$-free basis of V and $S = \{1, 2, \cdots, N\}$ be the index set. For each ordered index subset $I = \{i_1, i_2, \cdots, i_p\} \subset S$, we define e_I an element of $\bigwedge^p V$ as

$$e_I = \sum_{\sigma\in\mathfrak{S}_p} (\operatorname{sgn}\sigma) e_{i_{\sigma(1)}} \otimes e_{i_{\sigma(2)}} \otimes \cdots \otimes e_{i_{\sigma(p)}}.$$

Note that the set $\{e_I\}_{1\leqslant i_1<i_2<\cdots<i_p\leqslant N}$ provides a basis of $\bigwedge^p V$. Let E_{IJ} be an element of $\mathrm{End}(\bigwedge^p V)$ satisfying $E_{IJ}e_K = \delta_{JK}e_I$. Set

$$T_p(u) = \overset{p}{T}\left(u + \frac{p-1}{2}\overline{h}\right) \cdots \overset{2}{T}\left(u - \frac{p-3}{2}\overline{h}\right) \overset{1}{T}\left(u - \frac{p-1}{2}\overline{h}\right) A_p.$$

By lemma B.3, we see that $T_p(u)$ is the element of $\mathrm{End}(\bigwedge^p V)$. More precisely, we have the following lemma.

Lemma B.8.

$$T_p(u)e_J = \sum_I (q\text{-det}\, T_{IJ}(u)) e_I \qquad \text{or equivalently} \qquad T_p(u) = \sum_{I,J} (q\text{-det}\, T_{IJ}(u)) E_{IJ}.$$

One can prove this lemma by using corollary B.6.

Fix $p, q \in \mathbb{Z}_{>0}$ such that $p + q = N$. Regarding both $\bigwedge^N V$ and $\bigwedge^p V \otimes \bigwedge^q V$ as subspaces of $V^{\otimes N}$, one can easily express $e_S \in \bigwedge^N V$ by linear combinations of $e_I \otimes e_J \in \bigwedge^p V \otimes \bigwedge^q V$ as follows.

Lemma B.9.

$$e_S = \sum_{I\cup J=S;\#I=p} (-1)^{\frac{1}{2}p(p+1)+|I|} e_I \otimes e_J$$

where $|I| = \sum_{j=1}^{p} i_j$ for $I = \{i_1, i_2, \cdots, i_p\}$.

Combining lemma B.8 and lemma B.9, we obtain the Laplace expansion of the T-matrix.

Proposition B.10 (quantum Laplace expansion). For each $I, J \subset S$ with $\#I = p, \#J = q$, we have

$$(q\text{-det}\, T(u))\delta_{I\cup J,S}\delta_{I\cap J,\phi} = \sum_{K\cup L=S;\#K=p} (-1)^{|I|+|K|}\, q\text{-det}\, T_{IK}\left(u + \frac{q}{2}\hbar\right) \times q\text{-det}\, T_{JL}\left(u - \frac{p}{2}\hbar\right).$$

Specializing to $p = 1$ or $q = 1$ we obtain the quantum minor expansion of the T-matrix. Namely, set $S^{(i)} = S \setminus \{i\}$ and

$$\widetilde{T}(u) = (\tilde{t}_{ij}(u))_{1\leqslant i,j\leqslant N} \qquad \tilde{t}_{ij}(u) = (-1)^{i+j}\, q\text{-det}\, T_{S^{(j)},S^{(i)}}(u).$$

Then we get

Corollary B.11.

$$T\left(u + \frac{N-1}{2}\hbar\right)\widetilde{T}\left(u - \frac{1}{2}\hbar\right) = {}^t\widetilde{T}\left(u + \frac{1}{2}\hbar\right){}^tT\left(u - \frac{N-1}{2}\hbar\right) = (q\text{-det}\, T(u))I$$

where the superscript t denotes the transpose of the matrix.

B.2. Gauss decomposition of the T-matrix

In this subappendix, we explicitly construct the Gauss decomposition of $T(u)$ in terms of its quantum minors. Let

$$T(u) = \begin{pmatrix} 1 & & & 0 \\ f_{2,1}(u) & \ddots & & \\ & \ddots & \ddots & \\ f_{N,1}(u) & & f_{N,N-1}(u) & 1 \end{pmatrix} \begin{pmatrix} k_1(u) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & k_N(u) \end{pmatrix} \times \begin{pmatrix} 1 & e_{1,2}(u) & & e_{1,N}(u) \\ & \ddots & \ddots & \\ & & \ddots & e_{N-1,N}(u) \\ 0 & & & 1 \end{pmatrix}$$

be the Gauss decomposition of $T(u) = (t_{ij}(u))$.

Lemma B.12.

$$t_{i,j}(u) = \begin{cases} \sum_{l<i} f_{i,l}(u)k_l(u)e_{l,j}(u) + k_i(u)e_{i,j}(u) & i < j \\ \sum_{l<i} f_{i,l}(u)k_l(u)e_{l,i}(u) + k_i(u) & i = j \\ \sum_{l<j} f_{i,l}(u)k_l(u)e_{l,j}(u) + f_{i,j}(u)k_i(u) & i > j. \end{cases}$$

For $1 \leqslant p, q \leqslant N$, let us define $T_{p,q}(u)$ submatrices of $T(u)$ as follows.

Definition B.2. (i) $p = q$:

$$T_{p,p}(u) = \big(t_{i,j}(u)\big)_{1\leqslant i,j\leqslant p}$$

(ii) $p < q$:

$$T_{p,q}(u) = \begin{pmatrix} t_{1,1}(u) & \dots & t_{1,p-1}(u) & t_{1,q}(u) \\ \vdots & & \vdots & \vdots \\ t_{p-1,1}(u) & \dots & t_{p-1,p-1}(u) & t_{p-1,q}(u) \\ t_{p,1}(u) & \dots & t_{p,p-1}(u) & t_{p,q}(u) \end{pmatrix}$$

(iii) $p > q$:

$$T_{p,q}(u) = \begin{pmatrix} t_{1,1}(u) & \dots & t_{1,q-1}(u) & t_{1,q}(u) \\ \vdots & & \vdots & \vdots \\ t_{q-1,1}(u) & \dots & t_{q-1,q-1}(u) & t_{q-1,q}(u) \\ t_{p,1}(u) & \dots & t_{p,q-1}(u) & t_{p,q}(u) \end{pmatrix}.$$

Using lemma B.12, we can explicitly describe the Gauss decomposition of $T_{p,q}(u)$ as follows.

Lemma B.13. (i) $p = q$:

$$T_{p,p}(u) = \begin{pmatrix} 1 & & & 0 \\ f_{2,1}(u) & \ddots & & \\ \vdots & & \ddots & \\ f_{p,1}(u) & \cdots & f_{p,p-1}(u) & 1 \end{pmatrix} \begin{pmatrix} k_1(u) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ 0 & & & k_p(u) \end{pmatrix}$$

$$\times \begin{pmatrix} 1 & e_{1,2}(u) & \cdots & e_{1,p}(u) \\ & \ddots & \ddots & \vdots \\ & & \ddots & e_{p-1,p}(u) \\ 0 & & & 1 \end{pmatrix}$$

(ii) $p < q$:

$$T_{p,q}(u) = \begin{pmatrix} 1 & & & 0 & 0 \\ f_{2,1}(u) & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ f_{p-1,1}(u) & \cdots & f_{p-1,p-2}(u) & 1 & \\ f_{p,1}(u) & \dots\dots & \dots\dots & f_{p,p-1}(u) & 1 \end{pmatrix}$$

$$\times \begin{pmatrix} k_1(u) & & & 0 \\ & \ddots & & \\ & & \ddots & \\ & & & k_{p-1}(u) & \\ 0 & & & & k_p(u)e_{p,q}(u) \end{pmatrix}$$

$$\times\begin{pmatrix} 1 & e_{1,2}(u) & \cdots & e_{1,p-1}(u) & e_{1,q}(u) \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & e_{p-2,p-1}(u) & \vdots \\ 0 & & & 1 & e_{p-1,q}(u) \\ 0 & & & & 1 \end{pmatrix}$$

(iii) $p > q$:

$$T_{p,q}(u) = \begin{pmatrix} 1 & & & 0 & 0 \\ f_{2,1}(u) & \ddots & & & \\ \vdots & \ddots & \ddots & & \\ f_{q-1,1}(u) & \cdots & f_{q-1,q-2}(u) & 1 & \\ f_{p,1}(u) & \cdots & \cdots & f_{p,q-1}(u) & 1 \end{pmatrix}$$

$$\times\begin{pmatrix} k_1(u) & & & & 0 \\ & \ddots & & & \\ & & \ddots & & \\ & & & k_{q-1}(u) & \\ 0 & & & & f_{p,q}(u)k_q(u) \end{pmatrix}$$

$$\times\begin{pmatrix} 1 & e_{1,2}(u) & \cdots & e_{1,q-1}(u) & e_{1,q}(u) \\ & \ddots & \ddots & \vdots & \vdots \\ & & \ddots & e_{q-2,q-1}(u) & \vdots \\ 0 & & & 1 & e_{q-1,q}(u) \\ 0 & & & & 1 \end{pmatrix}.$$

Let

$$T_{p,q}(u) = F_{p,q}(u)K_{p,q}(u)E_{p,q}(u)$$

be the Gauss decomposition of $T_{p,q}(u)$ and set $r = \min\{p,q\}$. Comparing the (r,r) component of the formula

$$F_{p,q}(u)^{-1} = K_{p,q}(u)E_{p,q}(u)T_{p,q}(u)^{-1}$$

on both sides together with lemma B.13, we obtain the following.

Lemma B.14.

(i) $$k_p(u) = \frac{1}{\left[T_{p,p}(u)^{-1}\right]_{p,p}}$$

(ii) $$e_{p,q}(u) = \left[T_{p,p}(u)^{-1}\right]_{p,p} \frac{1}{\left[T_{p,q}(u)^{-1}\right]_{p,p}}$$

(iii) $$f_{p,q}(u) = \frac{1}{\left[T_{p,q}(u)^{-1}\right]_{q,q}} \left[T_{q,q}(u)^{-1}\right]_{q,q}$$

where $\left[T_{p,q}(u)^{-1}\right]_{a,b}$ signifies the (a,b) component of the matrix $T_{p,q}(u)^{-1}$.

Set

$$\Delta_{p,q}(u) := q\text{-det}\,.T_{p,q}(u) \qquad \Delta_p(u) := q\text{-det}\,.T_{p,p}(u).$$

Since we can express the matrix components of $T_{p,q}(u)^{-1}$ by their quantum minors lemma B.11, combining these with lemma B.14, we obtain the following results.

Theorem B.15.

(i) $$k_p(u) = \Delta_p\left(u - \frac{p-1}{2}\bar{h}\right)\Delta_{p-1}\left(u - \frac{p}{2}\bar{h}\right)^{-1}$$

(ii) $$e_{p,q}(u) = \Delta_p\left(u - \frac{p-1}{2}\bar{h}\right)^{-1}\Delta_{p,q}\left(u - \frac{p-1}{2}\bar{h}\right)$$

(iii) $$f_{p,q}(u) = \Delta_{p,q}\left(u - \frac{q-1}{2}\bar{h}\right)\Delta_q\left(u - \frac{q-1}{2}\bar{h}\right)^{-1}.$$

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